Chapter 4

On Parikh Matrices of words over formal alphabet

In 1966, R.J.Parikh introduced a notion called Parikh Mapping [69]. This notion is an important tool in the theory of formal language. With the help of this tool properties of words can be expressed numerically. In 2001 Mateescu et al. [61] introduced the notion of Parikh matrix. As mentioned earlier in chapter 1, a word is a finite or infinite sequence of symbols taken from a finite set called alphabet. This alphabet is an ordered alphabet. With every word over an ordered alphabet, a Parikh Matrix can be associated and it is a triangular matrix. All the entries of the main diagonal of this matrix is 1 and every entry below the main diagonal has the value 0 but the entries above the main diagonal provide information on the number of certain sub-words in the word . An interesting aspect of the Parikh Matrix is that it has the classical Parikh vector as the second diagonal above the main diagonal. As such Parikh Matrix of a word gives information about Parikh vector of the same. But Parikh Matrix faces some challenging problems, such as it is not injective. Two words may have the same corresponding Parikh Matrix. But two words with the same Parikh vector have in many

cases different Parikh matrices and thus the Parikh matrix gives more information about a word than the Parikh vector does. Because of this advantageous role, Parikh Matrix has become a research interest in related fields in recent years. In recent decades many techniques have been developed to solve complex problems of words using Parikh Matrix. A few examples [60, 78, 62, 88, 59, 74, 75, 76, 79, 87, 32] are cited which have used subword occurrences and Parikh matrix for solving the problems of word.
Binary words are words made by $\{a, b\}$. There are many methods by which we can form the Parikh matrix of binary words, for example simple matrix product and use of various computational tools etc. Here, an algorithm is introduced for finding Parikh matrix of a binary word. With the help of this algorithm Parikh matrix of a binary word, however large it may be, can be found out. So, we are having various methods to find out the Parikh matrix of a binary word. The above algorithm is extended for ternary words over $\{a, b, c\}$ and for tertiary words over $\{a, b, c, d\}$.

An algorithm is introduced in the present chapter to find the word corresponding to a Parikh matrix. This algorithm helps us to find the binary word corresponding to a Parikh matrix. Again, we know that Parikh matrix is not injective i.e. corresponding to a Parikh matrix there may be more than one word, this property is known as M-ambiguity. The words corresponding to a single Parikh matrix are known as amiable words or M-ambiguous words. With the help of this algorithm all the M-ambiguous words corresponding to a 3×3 Parikh matrix can be found instantly. One has to just enter a 3×3 Parikh matrix. If the matrix is not a Parikh matrix then there will be simply no corresponding word. This algorithm is also extended over ternary alphabet.

In this chapter a notion is introduced regarding the representation of binary words in two dimensional fields. This notion gives an interesting point of view regarding M- ambiguity. This type of graphical representation is extended over ternary alphabet. A set of equations which gives the corresponding binary word from a given 3×3 Parikh matrix is also introduced. Sets of equations are also developed for ternary and tertiary words.

The chapter is organized as follows. Section 4.1 goes toward developing the algorithm for display Parikh Matrix of a sequence over binary alphabet, ternary words and tertiary words. Section 4.2 also gives algorithm which gives binary words and ternary words, (may be one word or more than one word due to M-ambiguity) corresponding to a Parikh matrix; In section 4.3, notion of graphical representation of words in two and three dimensions is given. Section 4.4 gives proposed equations to find out the binary, ternary and tertiary sequences corresponding to a given Parikh Matrix. Section 4.5 gives definition of Stepping distance on classes of M-ambiguous words. Section 4.6 gives some characteristics of Parikh matrices. Investigations over Ratio property are summarised in 4.7. In Section 4.8 the introduction of M-ambiguity Reduction factor is given and finally in section 4.9 the chapter is concluded.

4.1 Algorithm to display Parikh matrix corresponding to a Formal word:

There are many methods by which we can form the Parikh matrix of binary words, for example, simple matrix product, use of various tools of computing matrix product etc. Here respective algorithm for finding Parikh matrix of a binary ,ternary and tertiary word is introduced. With the help of this algorithm Parikh matrix of a binary word, however large it may be, can be found out.

4.1.1 Algorithm for a Binary sequence:

The following pseudo code gives instantly the Parikh matrix of a binary sequence [19].

Algorithm:

1. Initialise a word = 'w'
2. Set len = length of w
3. **For** $i = 0$ to len do
4. Calculate total number of a, ab in 'w'
5. Calculate total number of b in 'w'
6. **End**

   Create a matrix (m_{ij}) of order $M = (4)$
7. **For** $i = 0$ to M do
8. **For** $j = 0$ to M do
9. **If**$(i = j)$
10. $a_{ij} = 1$
11. **else If**$(i > j)$
12. $a_{ij} = 0$
13. **else**
14. **If**$(i = 0 \& j = 1)$
15. $a_{ij} =$ total no of $'a'$
16. **If**$(i = 0 \& j = 2)$

$a_{ij} =$ total no of $'ab'$

If$(i = 1 \& j = 2)$

$a_{ij} =$ total no of $'b'$

End

End

4.1.2 Application of the algorithm for Binary sequence:

Example 1. The binary word $\xi_1 = abab\underbrace{a\cdots a}_{10}\underbrace{b\cdots b}_{15}$ has the Parikh matrix

$$\Psi_{M_2}(\xi_1) = \begin{pmatrix} 1 & 12 & 183 \\ 0 & 1 & 17 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 2. The binary word $\xi_2 = abababab\underbrace{a\cdots a}_{30}\underbrace{b\cdots b}_{29}$ has the Parikh matrix

$$\Psi_{M_2}(\xi_2) = \begin{pmatrix} 1 & 34 & 996 \\ 0 & 1 & 33 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 3. The binary word $\xi_3 = \underbrace{b\cdots b}_{20}ababababababab\underbrace{a\cdots a}_{38}$ has the Parikh matrix

$$\Psi_{M_2}(\xi_3) = \begin{pmatrix} 1 & 45 & 28 \\ 0 & 1 & 27 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 4. The binary word $\xi = \underbrace{a\cdots a}_{10}bb\underbrace{a\cdots a}_{10}bb$ has the Parikh matrix

$$\Psi_{M_2}(\xi_4) = \begin{pmatrix} 1 & 20 & 60 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

4.1.3 Algorithm for a Ternary sequence:

Any word over the ternary alphabet has a unique Parikh Matrix. This matrix can be obtained by simple matrix product of the respective Parikh matrices for a, b, c . For example, the word $abbcac$ has the Parikh Matrix $\begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ which can be obtained from simple matrix product

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and this can be obtained by using the theory of Parikh Matrix which is

$$\Psi_{M_3}(w) = \begin{pmatrix} 1 & |w|_a & |w|_{ab} & |w|_{abc} \\ 0 & 1 & |w|_b & |w|_{bc} \\ 0 & 0 & 1 & |w|_c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $|w|_a$ denotes the number of scattered sub- word of a and $|w|_{ab}$ denotes the number of Scattered sub- word of ab and so on. But for larger words the above processes are cumbersome, to overcome this problem the following algorithm is introduced. This algorithm gives instantly the Parikh matrix of a ternary sequence [17]. Results are verified. The following pseudo code gives the Parikh matrix of a ternary sequence.

Algorithm:

1. Initialise a word = 'w'
2. Set len = length of w
3. **For** $i = 0$ to len do
4. Calculate total number of a, ab, abc in 'w'
5. Calculate total number of b, bc in 'w'
6. Calculate total number of c in 'w'
7. **End**

   Create a matrix (m_{ij}) of order $M = (4)$

8. **For** $i = 0$ to M do
9. **For** $j = 0$ to M do
10. **If**$(i = j)$
11. $a_{ij} = 1$
12. **else If**$(i > j)$
13. $a_{ij} = 0$
14. **else**
15. **If**$(i = 0 \& j = 1)$
16. $a_{ij} =$ total no of $'a'$
17. **If**$(i = 0 \& j = 2)$

18. $a_{ij} =$ total no of $'ab'$

19. **If**$(i = 0 \& j = 3)$

20. $a_{ij} =$ total no of $'abc'$

21. **If**$(i = 1 \& j = 2)$

22. $a_{ij} =$ total no of $'b'$

23. **If**$(i = 1 \& j = 3)$

24. $a_{ij} =$ total no of $'bc'$

25. **If**$(i = 2 \& j = 3)$

26. $a_{ij} =$ total no of $'c'$

27. **End**

28. **End**

4.1.4 Application of the algorithm for Ternary sequence:

Example 1. The ternary word $\xi_1 = \underbrace{a \cdots a}_{5} bc \underbrace{b \cdots b}_{5} \underbrace{c \cdots c}_{6}$

has the Parikh matrix $\Psi_{M_3}(\xi_1) = \begin{pmatrix} 1 & 5 & 30 & 185 \\ 0 & 1 & 6 & 37 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

Example 2. The ternary word $\xi_2 = abc \underbrace{a \cdots a}_{10} \underbrace{b \cdots b}_{10} \underbrace{c \cdots c}_{10} abc$

has the Parikh matrix $\Psi_{M_3}(\xi_2) = \begin{pmatrix} 1 & 12 & 123 & 1234 \\ 0 & 1 & 12 & 123 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

Example 3. The ternary word $\xi_3 = bb\underbrace{a\cdots a}_{50}\underbrace{b\cdots b}_{28}\underbrace{c\cdots c}_{10}abbcc$

has the Parikh matrix $\Psi_{M_3}(\xi_3) = \begin{pmatrix} 1 & 51 & 1502 & 17004 \\ 0 & 1 & 32 & 364 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Example 4. The ternary word $\xi_4 = \underbrace{a\cdots a}_{15}\underbrace{c\cdots c}_{10}ba\underbrace{b\cdots b}_{18}$

has the Parikh matrix $\Psi_{M_3}(\xi_4) = \begin{pmatrix} 1 & 16 & 303 & 0 \\ 0 & 1 & 19 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

4.1.5 Algorithm for a Tertiary sequence:

The following pseudo code gives the Parikh matrix of a tertiary sequence [18].

Algorithm:

1. Initialise a word = 'w'

2. Set len = length of w

3. **For** $i = 0$ to len do

4. Calculate total number of $a, ab, abc, abcd$ in 'w'

5. Calculate total number of b, bc, bcd in 'w'

6. Calculate total number of c, cd in 'w'

7. Calculate total number of d in 'w'

8. **End**

   Create a matrix (m_{ij}) of order $M = (5)$

For $i = 0$ to M do

  For $j = 0$ to M do

    If$(i = j)$

      $a_{ij} = 1$

    else If$(i > j)$

      $a_{ij} = 0$

    else

    If$(i = 0 \& j = 1)$

      $a_{ij} =$ total no of $'a'$

    If$(i = 0 \& j = 2)$

      $a_{ij} =$ total no of $'ab'$

    If$(i = 0 \& j = 3)$

      $a_{ij} =$ total no of $'abc'$

    If$(i = 0 \& j = 4)$

      $a_{ij} =$ total no of $'abcd'$

    If$(i = 1 \& j = 2)$

      $a_{ij} =$ total no of $'b'$

    If$(i = 1 \& j = 3)$

      $a_{ij} =$ total no of $'bc'$

    If$(i = 1 \& j = 4)$

29. a_{ij} = total no of $'bcd'$

30. **If**$(i = 2 \& j = 3)$

31. a_{ij} = total no of $'c'$

32. **If**$(i = 2 \& j = 4)$

33. a_{ij} = total no of $'cd'$

34. **If**$(i = 3 \& j = 4)$

35. a_{ij} = total no of $'d'$

36. **End**

37. **End**

4.1.6 Application of the algorithm for Tertiary sequence:

Example 1. The tertiary word $\varsigma_1 = abcd\underbrace{a\cdots a}_{100}\underbrace{b\cdots b}_{100}\underbrace{c\cdots c}_{100}\underbrace{d\cdots d}_{100}$

has the Parikh matrix $\Psi_{M_4}(\varsigma_1) = \begin{pmatrix} 1 & 101 & 10101 & 1010101 & 101010101 \\ 0 & 1 & 101 & 10101 & 1010101 \\ 0 & 0 & 1 & 101 & 10101 \\ 0 & 0 & 0 & 1 & 101 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Example 2. The tertiary word $\varsigma_2 = abcd\underbrace{a\cdots a}_{10}\underbrace{b\cdots b}_{10}\underbrace{c\cdots c}_{10}\underbrace{d\cdots d}_{10}abcd$

has the Parikh matrix $\Psi_{M_4}(\varsigma_2) = \begin{pmatrix} 1 & 12 & 123 & 1234 & 12345 \\ 0 & 1 & 12 & 123 & 1234 \\ 0 & 0 & 1 & 12 & 123 \\ 0 & 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Example 3. The tertiary word $\varsigma_3 = \underbrace{a\cdots a}_{11}\underbrace{b\cdots b}_{11}\underbrace{c\cdots c}_{11}\underbrace{d\cdots d}_{11}$

has the Parikh matrix $\Psi_{M_4}(\varsigma_3) = \begin{pmatrix} 1 & 11 & 121 & 1331 & 14641 \\ 0 & 1 & 11 & 121 & 1331 \\ 0 & 0 & 1 & 11 & 121 \\ 0 & 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Example 4. The tertiary word $\varsigma_4 = abd\underbrace{c\cdots c}_{25}\underbrace{a\cdots a}_{10}\underbrace{d\cdots d}_{15}\underbrace{b\cdots b}_{10}abcd$

has the Parikh matrix $\Psi_{M_4}(\varsigma_4) = \begin{pmatrix} 1 & 12 & 123 & 148 & 523 \\ 0 & 1 & 12 & 37 & 412 \\ 0 & 0 & 1 & 26 & 401 \\ 0 & 0 & 0 & 1 & 17 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

4.2 Algorithm to display M-ambiguous words corresponding to a Parikh matrix:

4.2.1 Algorithm for a Binary sequence:

A few methods are used for finding M-ambiguous words for binary words; this algorithm [19] is also an effort in this regard. With the help of this algorithm all the M-ambiguous words corresponding to a 3×3 Parikh matrix can be found instantly. One has to just enter a 3×3 Parikh matrix. If the matrix is not a Parikh matrix then there will be simply no corresponding word. The following pseudo code gives instantly the binary sequences corresponding to a Parikh matrix.

Algorithm:

1. Input a matrix $A_{i,j}$ of order 3×3

2. $l = (A_{0,1} + A_{1,2})$

3. Store $'l'$ time of ab in w and consider it as an initial word $A_{i,j}$.

4. Make a list of words L where each word is obtained from w. (w is also included in L)

5. $L = \{l_1, l_1, l_1, \cdots, l_{\text{length of } w}\}$, l_i= word obtained from w by removing the i^{th} letter from it.

6. len = Number of elements of (L)

7. For $i = 1$ to len do

8. W = word at i^{th} th position of (L)

9. For $j = 1$ to length of W do

10. Store W by removing character at j^{th} position

11. End

12. Remove all duplicate copies of words from (L)

13. Remove W from (L) if numbers of a in $W \neq A_{0,1}$ and numbers of b in $W \neq A_{1,2}$

14. If it is removed then update i by $i - 1$

15. update len = Number of elements of (L)

16. if all the words of (L) is of length l display the words

17. End

18. For $k = 1$ to len do

19. Perform Algorithm 4.1.1 with each word at k^{th} position

20. It gives matrix $X_{i,j}$ of size 3×3 display the matrix

21. Check each of $X_{i,j}$ with $A_{i,j}$

22. If it matches then display *YES* otherwise display *NO*.

23. End.

4.2.2 Application of the algorithm for Binary sequence:

(a) Let $\Sigma = \{a < b\}$ and the Parikh matrix be $\Psi_{M_2}(\xi_1) = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, the set of amiable words having this Parikh matrix is $C = \{ababba, abbaab, baabab\}$

(b) Let $\Sigma = \{a < b\}$ and the Parikh matrix be $\Psi_{M_2}(\xi_1) = \begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, the set of amiable words having this Parikh matrix is

$C = \{ababaab, baaabab, aabbaba, abaabba\}$.

(c) Let $\Sigma = \{a < b\}$ and the Parikh matrix be $\Psi_{M_2}(\xi_1) = \begin{pmatrix} 1 & 7 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, the set of amiable words having this Parikh matrix is

$C = \{aabababaaa, abaabaabaa, ababaaaaba, baaaababaa, baaabaaaba,$

$baabaaaaab, aabbaaabaa, abaaabbaaaabbaaaaaab, aaabbbaaaa\}$.

(d) Let $\Sigma = \{a < b\}$ and the Parikh matrix be $\Psi_{M_2}(\xi_1) = \begin{pmatrix} 1 & 5 & 9 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$, the set of amiable words having this Parikh matrix is

$C = \{bbaabababa, bbabaabaab, babbaababa, babbabaaab, bababbaaba,$

$babababbaa, abbbabaaba, abbabbabaa, bbbaaaabab, bbaabbaaab,$

$bbabaaabba, baabbbabaa, abbbbaaaab, abbbaabbaa, ababbbbaaa, bbaaabbbaa\}$.

4.2.3 Algorithm for a Ternary sequence:

The following pseudo code gives instantly the ternary sequences corresponding to a Parikh matrix.

Algorithm:

1. Input a matrix $A_{i,j}$ of order 4×4.
2. $l = (A_{0,1} + A_{1,2} + A_{2,3})$
3. Store $'l'$ time of abc in w and consider it as an initial word $A_{i,j}$.
4. Make a list of words L where each word is obtained from w. (w is also included in L)
5. $L = \{l_1, l_1, l_1, \cdots, l_{\textit{length of w}}\}$, l_i= word obtained from w by removing the i^{th} letter from it.
6. len = Number of elements of (L)
7. For $i = 1$ to len do
8. W = word at i^{th} position of L
9. For $j = 1$ to length of W do
10. Store W by removing character at j^{th} position
11. End
12. Remove all duplicate copies of words from L
13. Remove W from L if numbers of a in $W \neq A_{0,1}$ and numbers of b in $W \neq A_{1,2}$ and numbers of c in $W \neq A_{2,3}$
14. If it is removed then update i by $i - 1$
15. update len = Number of elements of (L)

16. if all the words of L is of length l display the words

17. End

18. For $k = 1$ to len do

19. Perform Algorithm 4.1.3 with each word at k^{th} position

20. It gives matrix $X_{i,j}$ of size 4×4

21. Check each of $X_{i,j}$ with $A_{i,j}$

22. If it matches then display *YES* otherwise display *NO*.

23. End.

4.2.4 Application of the algorithm for Ternary sequence:

Examples:

- Let $\Sigma = \{a < b < c\}$ and the Parikh matrix be $\Psi_{M_3}(\zeta_1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the set of amiable words having this Parikh matrix is $C = \{cbbac, cbbca, bccba\}$

- Let $\Sigma = \{a < b < c\}$ and the Parikh matrix be $\Psi_{M_3}(\zeta_2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the set of amiable words having this Parikh matrix is $C = \{bcbac, bcbca\}$

- Let $\Sigma = \{a < b < c\}$ and the Parikh matrix be $\Psi_{M_3}(\zeta_3) = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the set of amiable words having this Parikh matrix is $C = \{abac, abca\}$

- Let $\Sigma = \{a < b < c\}$ and the Parikh matrix be $\Psi_{M_3}(\zeta_3) = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the set of amiable words having this Parikh matrix is the singleton set $C = \{cabab\}$

4.3 Introduction of the notion of Graphical representation of Formal words:

4.3.1 Introduction of the notion of Two dimensional representation of Binary words:

The aim of this section is to represent binary words in a two dimensional area [19]. For this, two perpendicular axes x and y (say) intersecting at a point are drawn. The coordinate of the intersecting point is considered to be (0, 0). In the x-axis it is taken a's and in the y-axis b's. Now let us draw the graph of the word. Let us start from $(0, 0)$ i.e. the intersection point of the two axes and go on describing the word w as follows: If the first letter of the word is a then we move to $(1, 0)$, if the second letter is again a then we move to $(2, 0)$ but if the second letter is b then we move to $(1, 1)$. If the first letter of the word is b then we move to $(0, 1)$, and thus go on describing the path of the word in the two dimensional graph. Let $\Psi_{M_2}(w) = \begin{pmatrix} 1 & |w|_a & |w|_{ab} \\ 0 & 1 & |w|_b \\ 0 & 0 & 1 \end{pmatrix}$ be the Parikh matrix corresponding to the word w. Now we draw two line segments. One line segment is parallel to the y-axis through the point $(|w|_a, 0)$ another line segment is parallel to the x-axis through the point $(0, |w|_b)$. Thus we shall get a closed bounded area as shown in figure 4.1. This area is either a square or a rectangle depending upon $|w|_a = |w|_b$ or $|w|_a \neq |w|_b$. Now we divide the area into $|w|_a + |w|_b$ lines. We draw $|w|_a$ equidistant lines parallel to the y-axis and $|w|_b$ equidistant lines parallel to the x-axis. These lines divide the prescribed area into $|w|_a \times |w|_b$ squares. The line traced by the word divides the rectangle in two parts. The upper part is bounded by the lines y-axis, the line parallel to the x-axis through the point $(0, |w|_b)$, and the line traced by the word itself. The lower part is bounded by the lines x-axis, the line parallel to the y-axis through the point $(|w|_a, 0)$, and the line traced by the word itself. The line traced by the word can be named as Word Line. The numbers of squares on the upper part is exactly equal to $|w|_{ab}$ and the number of squares on the lower part helps us to add some property in the field of amiable words. One can see that all the amiable words have the same area covered. Conversely, the words, corresponding to the same area covered, are M-ambiguous words. So we draw a word in the two dimensional field in the same way defined above and with the same $|w|_a$ and $|w|_b$ if we can draw another word which have the same area covered then the two words will be amiable. For example let us draw the word $abbaab$ on two dimensional areas.

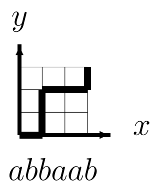
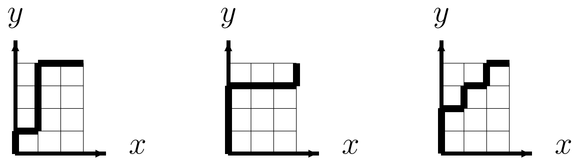


Figure 4.1: Graphical representation of $abbaab$

4.3.2 Result Analysis:

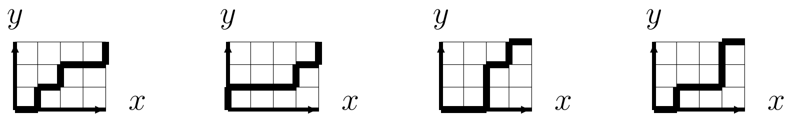
I. The two dimensional representation of the matrix $M_a = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ is given as follows:



1.$babbbaa$ 2.$bbbaaab$ 3.$bbababa$

Figure 4.2: Graphical representation of three M-ambiguous words

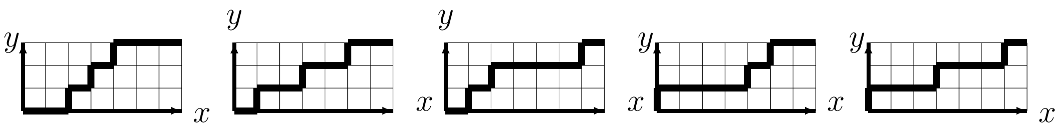
II. The two dimensional representation of the matrix $M_a = \begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ is given as follows:



1.$ababaab$ 2.$baaabab$ 3.$aabbaba$ 4.$abaabba$

Figure 4.3: Graphical representation of four M-ambiguous words

III. The two dimensional representation of the matrix $M_a = \begin{pmatrix} 1 & 7 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ is given as follows:



1.$aababababaaa$ 2.$abaabaabaa$ 3.$ababaaaaba$ 4.$baaaababaa$ 5.$baaabaaaba$

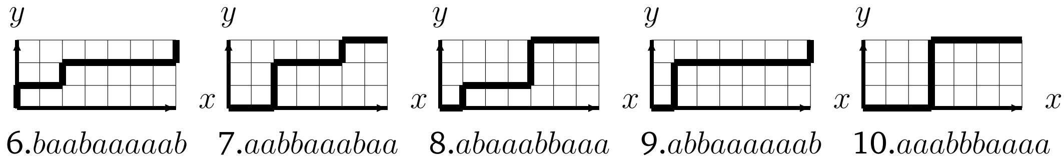


Figure 4.4: Graphical representation of ten M-ambiguous words

IV. The two dimensional representation of the matrix $M_a = \begin{pmatrix} 1 & 5 & 9 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$ is given as follows:

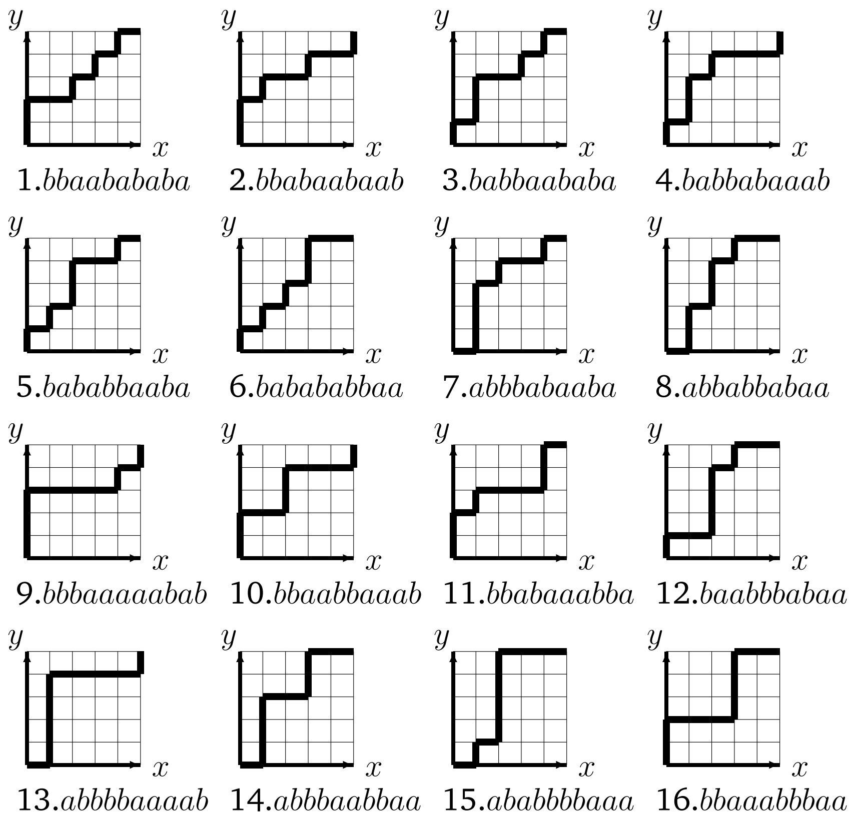


Figure 4.5: Graphical representation of sixteen M-ambiguous words

4.3.3 Introduction of the notion of Three dimensional representation of Ternary words:

As mentioned earlier, the ternary words are sequences made from the alphabet $\{a, b, c\}$. To represent ternary words in a three dimensional area one has to draw three perpendicular axes x, y and z (say) intersecting at a point [17]. It is considered that the coordinate of the intersecting point is (0,0,0) . In the x-axis we take a's, in the y-axis we take b's and in the z-axis we take c's. Now we draw the graph of the word. This graph can be named as Word Line. We start from (0,0,0) i.e. the intersecting point of the three axes and go on describing the word w as it is done in two dimensional case and thus get the path of the word in the three dimensional graph. Let $\Psi_{M_3}(w) = \begin{pmatrix} 1 & |w|_a & |w|_{ab} & |w|_{abc} \\ 0 & 1 & |w|_b & |w|_{bc} \\ 0 & 0 & 1 & |w|_c \\ 0 & 0 & 0 & 1 \end{pmatrix}$ be the Parikh matrix corresponding to the word w. Now we draw six plane segments. First three are $x = 0, y = 0, z = 0$, then one plane segment is perpendicular to the x-axis and its equation is $x = |w|_a$ another plane segment is perpendicular to the y-axis and its equation is $y = |w|_b$ another plane segment is perpendicular to the z-axis and its equation is $z = |w|_c$. Thus we shall get a closed bounded region. This region is either a cube or a cuboid depending upon $|w|_a = |w|_b = |w|_c$ or else. Now we divide the region into $|w|_a + |w|_b + |w|_c$ planes. We draw $|w|_a$ equidistant planes perpendicular to the x-axis, $|w|_b$ equidistant planes perpendicular to the y-axis, and $|w|_c$ equidistant planes perpendicular to the z-axis. These planes divide the prescribed region into $|w|_a \times |w|_b \times |w|_c$ cubes.

4.4 Set of equations to find Formal words corresponding to a Parikh matrix:

In this section it is tried to solve the problem of M-ambiguity by using algebraic equations. In this approach for each case of binary , ternary, tertiary ordered alphabets a set of equations is developed separately.

4.4.1 Set of equations to find Binary words corresponding to a Parikh matrix:

In [19] a set of equations corresponding to a binary word $\beta \in \Sigma^*$ has been introduced. Let $\Sigma = \{a, b\}$ be a binary ordered alphabet and $\beta \in \Sigma^*$ be a binary sequence. If $|w|_a = f$,$|w|_b = g$ then β can be represented in the following form: $\beta = a^{x_1}b^{y_1}a^{x_2}b^{y_2}\cdots a^{x_{f+g}}b^{y_{f+g}}$,the Parikh matrix $\Psi_{M_2}(\beta) = \begin{pmatrix} 1 & f & h \\ 0 & 1 & g \\ 0 & 0 & 1 \end{pmatrix}$ corresponds to this word if and only if

x_i = either 0 or 1 and

y_j= either 0 or 1,

is a solution of the following system of equations:

$$\sum_{i=1}^{f+g} x_i = f \tag{4.1}$$

$$\sum_{j=1}^{f+g} y_j = g \tag{4.2}$$

$$\sum_{i=1}^{f+g} x_i \sum_{j=i}^{f+g} y_j = h \tag{4.3}$$

For clear understanding we take the example of the following Parikh matrix. Example 1: Let $\Psi_{M_2}(\varsigma_1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

be a Parikh matrix. Then $\varsigma_1 \in \Sigma^*$ is a binary sequence corresponding to the above matrix. Here $f + g = 1 + 1 = 2$. So ς_1 can be represented in the following form:

$\varsigma_1 = a^{x_1} b^{y_1} a^{x_2} b^{y_2}$. The Parikh Matrix corresponds to this word if and only if

x_i = either 0 or 1,

y_j = either 0 or 1,

is a solution of the following system of equations:

$$\sum_{i=1}^{2} x_i = 1 \tag{4.4}$$

$$\sum_{j=1}^{2} y_j = 1 \tag{4.5}$$

$$\sum_{i=1}^{2} x_i \sum_{j=i}^{2} y_j = 0 \tag{4.6}$$

Now from(4.4) , $x_1 + x_2 = 1$

from (4.5), $y_1 + y_2 = 1$

from (4.6),

$x_1 (y_1 + y_2) + x_2 y_2 = 0.$

$\Rightarrow x_1(1) + x_2 y_2 = 0$[from (4.5)]

$\Rightarrow y_2 = 0$[using (4.4)]

$\therefore y_1 = 1$ again from (4.6) we have

$x_1(y_1 + y_2) + x_2 y_2 = 0$

$\Rightarrow x_1(1 + 0) + x_2 \cdot 0 = 0$

$\Rightarrow x_1 = 0$ So the word $\varsigma_1 = a^{x_1} b^{y_1} a^{x_2} b^{y_2}$ is $\varsigma_1 = a^0 b^1 a^1 b^0 = ba$.

Example 2: Let$\Psi_{M_2}(\varsigma_2) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

be a Parikh matrix. Here $|w|_a = 1$,$|w|_b = 1$. Then ς_2 is a binary sequence that corresponds to the above matrix. Here $f + g = 1 + 1 = 2$. So ς_2 can be represented in the following form:

$\varsigma_2 = a^{x_1} b^{y_1} a^{x_2} b^{y_2}$. The Parikh Matrix corresponds to this word if and only if

x_i = either 0 or 1,

y_j = either 0 or 1,

is a solution of the following system of equations:

$$\sum_{i=1}^{2} x_i = 1 \tag{4.7}$$

$$\sum_{j=1}^{2} y_j = 1 \tag{4.8}$$

$$\sum_{i=1}^{2} x_i \sum_{j=i}^{2} y_j = 1 \tag{4.9}$$

Now from (4.7), $x_1 + x_2 = 1$

from (4.8), $y_1 + y_2 = 1$

from (4.9), $x_1\left(y_1 + y_2\right) + x_2 y_2 = 1$

$\Rightarrow x_1(1) + x_2 y_2 = 1$ [from (4.8)]

$\Rightarrow$ either $x_1 = 1, x_2 = 0, y_2 = 0$ or $x_1 = 1, x_2 = 1, y_2 = 0$

if $y_2 = 0$ then from (4.8) we have

$y_1 = 1$

$\Rightarrow$ either $x_1 = 1, y_1 = 1$ or $x_2 = 1, y_2 = 1$

So the word $\varsigma_1 = a^{x_1} b^{y_1} a^{x_2} b^{y_2}$ is either $\varsigma_1 = a^1 b^1 a^0 b^0 = ab$ or $\varsigma_1 = a^0 b^0 a^1 b^1 = ab$.

This is how we can use the proposed set of equations to find the corresponding word from a 3×3 Parikh matrix.

4.4.2 Set of equations to find Ternary words corresponding to a Parikh matrix:

A set of equations has been developed for finding the ternary sequences corresponding to a given Parikh matrix for a ternary ordered alphabet [17]. The equations may have more than one solution and these solutions give a set of ternary amiable words. Let $\Sigma = \{a, b, c\}$ be a ternary ordered alphabet and ξ be a ternary sequence. If $|w|_a = l$,$|w|_b = m$,$|w|_c = n$ and then ξ can be represented in the following form:

$\xi = a^{x_1}b^{y_1}c^{z_1}a^{x_2}b^{y_2}c^{z_2} \cdots a^{x_{l+m+n}}b^{y_{l+m+n}}c^{z_{l+m+n}}$.

The Parikh Matrix $\Psi_{M_3}(\xi) = \begin{pmatrix} 1 & l & p & r \\ 0 & 1 & m & q \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}$ corresponds to this word if and only if

x_i= either 0 or 1,

y_j = either 0 or 1 and

z_k = either 0 or 1,

is a solution of the following system of equations:

$$\sum_{i=1}^{l+m+n} x_i = l \tag{4.10}$$

$$\sum_{j=1}^{l+m+n} y_j = m \tag{4.11}$$

$$\sum_{k=1}^{l+m+n} z_k = n \tag{4.12}$$

$$\sum_{i=1}^{l+m+n} x_i \sum_{j=i}^{l+m+n} y_j = p \tag{4.13}$$

$$\sum_{j=1}^{l+m+n} y_j \sum_{k=j}^{l+m+n} z_k = q \tag{4.14}$$

$$\sum_{i=1}^{l+m+n} x_i \sum_{j=i}^{l+m+n} y_j \sum_{k=j}^{l+m+n} z_k = r \tag{4.15}$$

Example 1: For clear understanding we take the example of the following Parikh matrix. Let $\Psi_{M_3}(\varsigma_1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

be a Parikh matrix. Here $|w|_a = 1$,$|w|_b = 1$,$|w|_c = 0$, and so on. Then $\varsigma_1 \in \Sigma^*$ is a ternary sequence corresponding to the above matrix. Here $l + m + n = 1 + 1 + 0 + = 2$. So ς_1 can be represented in the following form: $\varsigma = a^{x_1} b^{y_1} c^{z_1} a^{x_2} b^{y_2} c^{z_2}$. The Parikh Matrix corresponds to this word if and only if

x_i = either 0 or 1,

y_j = either 0 or 1 and

z_k = either 0 or 1,

is a solution of the following system of equations:

$$\sum_{i=1}^{l+m+n} x_i = 1 \tag{4.16}$$

$$\sum_{j=1}^{l+m+n} y_j = 1 \tag{4.17}$$

$$\sum_{k=1}^{l+m+n} z_k = 0 \tag{4.18}$$

$$\sum_{i=1}^{l+m+n} x_i \sum_{j=i}^{l+m+n} y_j = 0 \tag{4.19}$$

$$\sum_{j=1}^{l+m+n} y_j \sum_{k=j}^{l+m+n} z_k = 0 \tag{4.20}$$

$$\sum_{i=1}^{l+m+n} x_i \sum_{j=i}^{l+m+n} y_j \sum_{k=j}^{l+m+n} z_k = 0 \tag{4.21}$$

Now from (4.16), $x_1 + x_2 = 1$

from (4.17), $y_1 + y_2 = 1$

from (4.19),

$x_1(y_1 + y_2) + x_2y_2 = 0$

$\Rightarrow x_1(1) + x_2y_2 = 0$[from (4.17)]

$\Rightarrow y_2 = 0$[using (4.16)]

$\therefore y_1 = 1$ again from (4.19) we have

$x_1(y_1 + y_2) + x_2y_2 = 0$

$\Rightarrow x_1(1 + 0) + x_2 \cdot 0 = 0$

$\Rightarrow x_1 = 0$ So the word $\varsigma_1 = a^{x_1}b^{y_1}c^{z_1}a^{x_2}b^{y_2}c^{z_2}$ is $\varsigma_1 = a^0b^1c^0a^1b^0c^0 = ba$

Example 2: For the Parikh matrix

$$\Psi_{M_3}(\varsigma_2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here $|w|_a = 1$,$|w|_b = 1$,$|w|_c = 1$ and so on. Then $\varsigma_2 \in \Sigma^*$ is a ternary sequence corresponds to the above matrix. Here $l + m + n = 1 + 1 + 1 = 3$. So ς_2 can be represented in the following form: $\varsigma_2 = a^{x_1}b^{y_1}c^{z_1}a^{x_2}b^{y_2}c^{z_2}a^{x_3}b^{y_3}c^{z_3}$

The Parikh Matrix corresponds to this word if and only if

x_i = either 0 or 1,

y_j = either 0 or 1,

z_k = either 0 or 1,

is a solution of the system of equations $($(4.10)$, \cdots,$ (4.15)$)$. We get the solution of the equations as $x_3 = 1, y_2 = 1$ and $z_1 = 1$ and remaining all others are 0. Then the word is $\varsigma_2 = a^{x_1}b^{y_1}c^{z_1}a^{x_2}b^{y_2}c^{z_2}a^{x_3}b^{y_3}c^{z_3}$

$\Rightarrow \varsigma_2 = a^0b^0c^1a^0b^1c^0a^1b^0c^0 = cba.$

This is how we can use the proposed set of equations to find the corresponding word from a 4×4 Parikh matrix.

4.4.3 Set of equations to find Tertiary words corresponding to a Parikh matrix:

A set of equations for finding words over tertiary alphabet from the respective Parikh matrix has been introduced in [18]. This set of equations for finding the tertiary sequences corresponds to a given 5×5 Parikh matrix. Let$\Sigma = a, b, c, d$ be a tertiary ordered alphabet and

$$\Psi_{M_4}(\varsigma) = \begin{pmatrix} 1 & A & E & H & J \\ 0 & 1 & B & F & I \\ 0 & 0 & 1 & C & G \\ 0 & 0 & 0 & 1 & D \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{———(I)}$$

be a Parikh matrix i.e. $|w|_a = A$,$|w|_b = B$, $|w|_c = C$,$|w|_d = D$ and so on. Then $\varsigma \in \Sigma^*$ is a tertiary sequence that corresponds to the above matrix if ς can be represented in the following form:

$\varsigma = a^{x_1}b^{y_1}c^{z_1}d^{t_1}a^{x_2}b^{y_2}c^{z_2}d^{t_2}\cdots a^{x_{A+B+C+D}}b^{y_{A+B+C+D}}c^{z_{A+B+C+D}}d^{t_{A+B+C+D}}$. The Parikh Matrix corresponds to this word if and only if

x_i= either 0 or 1,

y_j = either 0 or 1,

z_k = either 0 or 1 and

t_l = either 0 or 1,

is a solution of the following system of equations:

$$\sum_{i=1}^{A+B+C+D} x_i = A \tag{4.22}$$

$$\sum_{j=1}^{A+B+C+D} y_j = B \tag{4.23}$$

$$\sum_{k=1}^{A+B+C+D} z_k = C \tag{4.24}$$

$$\sum_{l=1}^{A+B+C+D} t_l = D \tag{4.25}$$

$$\sum_{i=1}^{A+B+C+D} x_i \sum_{j=i}^{A+B+C+D} y_j = E \tag{4.26}$$

$$\sum_{j=1}^{A+B+C+D} y_j \sum_{k=j}^{A+B+C+D} z_k = F \tag{4.27}$$

$$\sum_{k=1}^{A+B+C+D} z_k \sum_{l=k}^{A+B+C+D} t_l = G \tag{4.28}$$

$$\sum_{i=1}^{A+B+C+D} x_i \sum_{j=i}^{A+B+C+D} y_j \sum_{k=j}^{A+B+C+D} z_k = H \tag{4.29}$$

$$\sum_{j=1}^{A+B+C+D} y_j \sum_{k=j}^{A+B+C+D} z_k \sum_{l=k}^{A+B+C+D} t_l = I \tag{4.30}$$

$$\sum_{i=1}^{A+B+C+D} x_i \sum_{j=i}^{A+B+C+D} y_j \sum_{k=j}^{A+B+C+D} z_k \sum_{l=k}^{A+B+C+D} t_l = J \tag{4.31}$$

Example 1: For clear understanding we take the example of the following Parikh matrix. Let $\Psi_{M_4}(\varsigma_1) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

be a Parikh matrix. Here $|w|_a = 1$,$|w|_b = 1$,$|w|_c = 0$,$|w|_d = 0$ and so on. Then $\varsigma_1 \in \Sigma^*$ is a tertiary sequence that corresponds to the above matrix. Here $A + B + C + D = 1 + 1 + 0 + 0 = 2$. So ς_1 can be represented in the following form:

$\varsigma = a^{x_1} b^{y_1} c^{z_1} d^{t_1} a^{x_2} b^{y_2} c^{z_2} d^{t_2}$. The Parikh Matrix corresponds to this word if and only if

x_i = either 0 or 1,

y_j = either 0 or 1,

z_k = either 0 or 1 and

t_l = either 0 or 1,

is a solution of the following system of equations:

$$\sum_{i=1}^{2} x_i = 1 \tag{4.32}$$

$$\sum_{j=1}^{2} y_j = B \tag{4.33}$$

$$\sum_{k=1}^{2} z_k = 0 \tag{4.34}$$

$$\sum_{l=1}^{2} t_l = 0 \tag{4.35}$$

$$\sum_{i=1}^{2} x_i \sum_{j=i}^{2} y_j = 0 \tag{4.36}$$

$$\sum_{j=1}^{2} y_j \sum_{k=j}^{2} z_k = 0 \tag{4.37}$$

$$\sum_{k=1}^{2} z_k \sum_{l=k}^{2} t_l = 0 \tag{4.38}$$

$$\sum_{i=1}^{2} x_i \sum_{j=i}^{2} y_j \sum_{k=j}^{2} z_k = 0 \tag{4.39}$$

$$\sum_{j=1}^{2} y_j \sum_{k=j}^{2} z_k \sum_{l=k}^{2} t_l = 0 \tag{4.40}$$

$$\sum_{i=1}^{2} x_i \sum_{j=i}^{2} y_j \sum_{k=j}^{2} z_k \sum_{l=k}^{2} t_l = 0 \tag{4.41}$$

Now from (4.32), $x_1 + x_2 = 1$

from (4.33), $y_1 + y_2 = 1$

from (4.36),

$x_1(y_1 + y_2) + x_2 y_2 = 0$

$\Rightarrow x_1(1) + x_2 y_2 = 0$[from (4.33)]

$\Rightarrow y_2 = 0$[using (4.32)]

$\therefore y_1 = 1$ again from (4.36) we have

$x_1(y_1 + y_2) + x_2 y_2 = 0$

$\Rightarrow x_1(1 + 0) + x_2 \cdot 0 = 0$

$\Rightarrow x_1 = 0$ So the word $\varsigma_1 = a^{x_1} b^{y_1} c^{z_1} d^{t_1} a^{x_2} b^{y_2} c^{z_2} d^{t_2}$ is $\varsigma_1 = a^0 b^1 c^0 d^0 a^1 b^0 c^0 d^0 = ba$

Example 2: For the Parikh matrix $\Psi_{M_4}(\varsigma_2) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Here $|w|_a = 1$,$|w|_b = 1$,$|w|_c = 1$,$|w|_d = 1$ and so on. Then $\varsigma_2 \in \Sigma^*$ is a tertiary sequence corresponding to the above matrix. Here $A + B + C + D = 1 + 1 + 1 + 1 = 4$. So ς_2 can be represented in the following form:

$\varsigma_2 = a^{x_1} b^{y_1} c^{z_1} d^{t_1} a^{x_2} b^{y_2} c^{z_2} d^{t_2} a^{x_3} b^{y_3} c^{z_3} d^{t_3} a^{x_4} b^{y_4} c^{z_4} d^{t_4}$

The Parikh Matrix corresponds to this word if and only if

x_i = either 0 or 1,

y_j = either 0 or 1,

z_k = either 0 or 1 and

t_l = either 0 or 1,

is a solution of the system of equations $((4.32), \cdots, (4.41))$. We get the solution of the equations as $x_4 = 1, y_3 = 1, z_2 = 1$ and $t_1 = 1$ And remaining all others is 0. Then the word is

$\varsigma_2 = a^{x_1}b^{y_1}c^{z_1}d^{t_1}a^{x_2}b^{y_2}c^{z_2}d^{t_2}a^{x_3}b^{y_3}c^{z_3}d^{t_3}a^{x_4}b^{y_4}c^{z_4}d^{t_4}$

$\Rightarrow \varsigma_2 = a^0b^0c^0d^1a^0b^0c^1d^0a^0b^1c^0d^0a^1b^0c^0d^0 = dcba$

This is how we can use the proposed set of equations to find the corresponding word from a 5×5 Parikh matrix.

4.5 Definition of Stepping distance on classes of M- ambiguous words:

Comparing M- ambiguous words by distance is an old process [6]. In this present study a new type of distance is introduced [17]. For convenience it is assumed that the symbols $a, b, c, d, \cdots$ are lying on a straight line. So to describe the distance between two symbols on the straight line we have to either step forward or step backward. For this reason this distance can be named as Stepping Distance. Using this notion of Stepping Distance we can compare M-ambiguous words.

4.5.1 Stepping distance on classes of M-ambiguous Binary words:

For binary ordered alphabet it is assumed that the symbols a, b are lying on a straight line. Using this notion of stepping distance we can compare M-ambiguous words. Let $\alpha = a_1a_2a_3 \cdots a_n$; $a_i \in \{a, b\}$ and $\beta = b_1b_2b_3 \cdots b_n$; $b_i \in \{a, b\}$

be two M- ambiguous words. The stepping distance on the class of M-ambiguous words over $\Sigma = \{a < b\}$ is defined as $d_S(\alpha, \beta) = \sum_{i=1}^{n}(a_i +_S b_i)$. Where $+_S : \Sigma \times \Sigma \longrightarrow \{0, 1\}$ is defined by

$a +_S a = 0, b +_S b = 0,$

$a +_S b = 1 = b +_S a,$

For example, the stepping distance $d_S(\alpha, \beta)$ between the following two amiable words $\alpha = babbaa, \beta = bbaaba$ over the Parikh matrix $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ is 4.

4.5.2 Stepping distance on classes of M-ambiguous Ternary words:

For convenience it is assumed that the symbols a, b, c are lying on a straight line [17]. Using this notion of stepping distance we can compare M-ambiguous words. Let $\alpha = a_1a_2a_3 \cdots a_n$; $a_i \in \{a, b, c\}$ and $\beta = b_1b_2b_3 \cdots b_n$; $b_i \in \{a, b, c\}$

be two M- ambiguous words. The stepping distance on the class of M-ambiguous words over $\Sigma = \{a < b < c\}$ is defined as $d_S(\alpha, \beta) = \sum_{i=1}^{n}(a_i +_S b_i)$. Where $+_S : \Sigma \times \Sigma \longrightarrow \{0, 1, 2\}$ is defined by

$a +_S a = 0, b +_S b = 0, c +_S c = 0,$

$a +_S b = 1 = b +_S a,$

$b +_S c = 1 = c +_S b$

$a +_S c = 2 = c +_S a$, For example, the stepping distance $d_S(\alpha, \beta)$ between the following two amiable words $\alpha = bcbac, \beta = bcbca$ over the Parikh matrix $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is 4.

4.5.3 Stepping distance on classes of M-ambiguous Tertiary words :

For convenience it is assumed that the symbols a, b, c, d are lying on a straight line. Using this notion of stepping distance we can compare M-ambiguous words. Let $\alpha = a_1a_2a_3\cdots a_n$; $a_i \in \{a, b, c, d\}$ and $\beta = b_1b_2b_3\cdots b_n$; $b_i \in \{a, b, c, d\}$

be two M- ambiguous words. The stepping distance on the class of M-ambiguous words over $\Sigma = \{a < b < c < d\}$ is defined as $d_S(\alpha, \beta) = \sum_{i=1}^{n}(a_i +_S b_i)$. Where $+_S : \Sigma \times \Sigma \longrightarrow \{0, 1, 2, 3\}$ is defined by

$a +_S a = 0, b +_S b = 0, c +_S c = 0, d +_S d = 0,$

$a +_S b = 1 = b +_S a,$

$b +_S c = 1 = c +_S b,$

$c +_S d = 1 = d +_S c,$

$a +_S c = 2 = c +_S a,$

$b +_S d = 2 = d +_S b,$

$a +_S d = 3 = d +_S a,$

For example, the stepping distance $d_S(\alpha, \beta)$ between the following two M- ambiguous words $\alpha = abcddcbadcba, \beta = abdccbddacba$ of the Parikh matrix $\Psi_{M_4}(\varsigma_4) = \begin{pmatrix} 1 & 3 & 4 & 4 & 4 \\ 0 & 1 & 3 & 4 & 4 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ is 12.

4.6 A few observations on Parikh matrices:

Theorem: If the Parikh Matrix $\Psi_{M_3}(\xi) = \begin{pmatrix} 1 & l & p & r \\ 0 & 1 & m & q \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}$ corresponds to a ternary sequence then $r \in [0, l.m.n]$.

Proof: Let the Parikh Matrix $\Psi_{M_3}(\xi) = \begin{pmatrix} 1 & l & p & r \\ 0 & 1 & m & q \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}$ corresponds to a word $\xi \in \Sigma^*$. Where $\Sigma = \{a, b, c\}$ is a ternary alphabet. ξ can be represented in the following form: $\xi = a^{x_1}b^{y_1}c^{z_1}a^{x_2}b^{y_2}c^{z_2}\cdots a^{x_l}b^{y_m}c^{z_n}$. Now by using equations (4.10), (4.11), (4.12) and (4.15),

$l.m.n$

$= \sum_{i=1}^{l+m+n} x_i \sum_{j=1}^{l+m+n} y_j \sum_{k=1}^{l+m+n} z_k$

$> \sum_{i=1}^{l+m+n} x_i \sum_{j=i}^{l+m+n} y_j \sum_{k=j}^{l+m+n} z_k = r$

$\Rightarrow r \in [0, l.m.n]$.

But the converse is not always true; this can be illustrated by the example of matrix $\Psi_{M_3}(\xi) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We can see that this upper triangular matrix does not have any corresponding ternary sequence.

Remark: In the Parikh matrix $\Psi_{M_3}(w) = \begin{pmatrix} 1 & |w|_a & |w|_{ab} & |w|_{abc} \\ 0 & 1 & |w|_b & |w|_{bc} \\ 0 & 0 & 1 & |w|_c \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for a ternary sequence w whenever $|w|_{abc} = 0$ then either $|w|_{ab} = 0$ or $|w|_{bc} = 0$

Example: The Parikh matrix of the ternary sequence bca is $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

It is seen that $|w|_{abc} = 0$ this matrix has a corresponding ternary sequence as here $|w|_{ab} = 0$, again the Parikh matrix corresponding to the ternary sequence cab is $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

It is seen that $|w|_{abc} = 0$ this matrix also has a corresponding ternary sequence as here $|w|_{bc} = 0$.

Remark: From [61] we have the result that the entries $m_{i,j+1}, 1 \leq i \leq j \leq k$ in a Parikh matrix $\Psi_{M_k}(w)$ satisfy the inequality $m_{i,j+1} \leq m_{i,j} \leq m_{i+1,j+1}$. Here for ternary sequence $m_{i,j+1} \leq m_{i,j} \leq m_{i+1,j+1}$, $m_{i,j+1} \leq m_{i,j} \leq m_{i+1,j+1}$, $m_{i,j+1} \leq m_{i,j} \leq m_{i+1,j+1}$.

Remark: In the paper [32] it is given that for $w \in Z^* = \{a, b, c\}^*$ using the notation $\Psi_{M_3}(W) = \begin{pmatrix} 1 & A & E & x \\ 0 & 1 & B & F \\ 0 & 0 & 1 & C \\ 0 & 0 & 0 & 1 \end{pmatrix}$

it can be obtained that $AF + CE - ABC \leq x \leq \frac{EF}{B}$. We can prove the same by using the equations (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15).

More Examples:

- (a) Let $\Sigma = \{a < b < c\}$ and the Parikh matrix be

$$\Psi_{M_3}(\zeta_1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the set of corresponding amiable words having this Parikh matrix is $C = \{cbbac, cbbca, bccba\}$. It can be seen that the inequality $AF + CE - ABC \leq x \leq \frac{EF}{B}$ is satisfied.

- For the following matrices
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
there is no corresponding ternary word. It can be verified from the inequality $AF + CE - ABC \leq x \leq \frac{EF}{B}$ given in [32]. It is seen that the right hand side of the inequality is not satisfied by these matrices. From this it can be understood that every upper triangular matrix having the diagonal entries 1 cannot be a Parikh matrix.

4.7 Ratio property and Weak ratio property of sequences:

In this section ratio property and weak ratio property of words are analysed. Concept of ratio property and weak ratio property are extended for n^{th} order alphabet. A relationship of ratio property with M- ambiguity is established. Various lemmas already proved about ratio property over ternary alphabet are proposed for tertiary alphabets. M-ambiguous words are formed by concatenating words satisfying ratio property. Various methods are used to find M-ambiguous words and to solve the problems of M-ambiguity. Ratio property and weak-ratio property introduced in [87] also throw light in this direction. The weak-ratio property is used in [57] to prove some interesting lemma regarding the commutativity of the Parikh matrix of two words over binary and ternary alphabets. In the paper [89] the weak-ratio property is extended to binary arrays and new extensions of this property namely row-ratio property and column-ratio property are introduced. Various results are also investigated in that field. Weak-ratio property is also used in [88] with association with Istrail morphism to explore a new area of investigation of morphic images of words. In this section ratio property and weak ratio property have been studied in a generalized approach [15]. Words over tertiary alphabet are discussed in the light of ratio property. It is seen that two words satisfying ratio property make M-ambiguous words when they are allied in certain ways.

The section is organized as follows. The subsection 4.7.1 presents the development of the generalized definition of ratio property. Section 4.7.2 gives generalization of weak-ratio property. In Section 4.7.3 relationship of ratio property with M-ambiguity is discussed and various lemmas already proved for ternary sequences in [87] are investigated for tertiary sequences.

4.7.1 Generalization of Ratio property:

Let $\Sigma = \{a_1 < a_2 < a_3 < \cdots < a_n\}$ be an n^{th} ordered alphabet. Two words w_1, w_2 over $\Sigma = \{a_1 < a_2 < a_3 < \cdots < a_n\}$ are said to satisfy the ratio property, written $w_1 \backsim_r w_2$

$$\text{if } \Psi_{M_4}(w_1) = \begin{pmatrix} 1 & p_1 & p_{1,2} & \dots & p_{1,n-2} & p_{1,n-1} & p_{1,n} \\ 0 & 1 & p_2 & \dots & p_{2,n-2} & p_{2,n-1} & p_{2,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & p_{n-1} & p_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & 1 & p_n \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}_{(n+1)\times(n+1)}$$

and

$$\Psi_{M_4}(w_2) = \begin{pmatrix} 1 & q_1 & q_{1,2} & \dots & q_{1,n-2} & q_{1,n-1} & q_{1,n} \\ 0 & 1 & q_2 & \dots & q_{2,n-2} & q_{2,n-1} & q_{2,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & q_{n-1} & q_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & 1 & q_n \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}_{(n+1)\times(n+1)}$$

satisfy the conditions $p_i = s \cdot q_i$, $(i = 1, 2, 3, \dots, n)$, $p_{i,i+1} = s \cdot q_{i,i+1}$, $(i = 1, 2, 3, \dots, n-1)$, $\dots$, $p_{i,i+2} = s \cdot q_{i,i+2}$, $(i = 1, 2)$ where s is a constant.

4.7.2 Generalization of Weak-ratio property:

Let $\Sigma = \{a_1 < a_2 < a_3 < \cdots < a_n\}$ be an n^{th} ordered alphabet. Two words w_1, w_2 over $\Sigma = \{a_1 < a_2 < a_3 < \cdots < a_n\}$ are said to satisfy the weak ratio property, written $w_1 \backsim_{wr} w_2$. if $\Phi_{M_4}(w_1) = \begin{pmatrix} p_1 & p_{1,2} & \dots & p_{1,n} \\ p_{2,1} & p_2 & \dots & p_{2,n} \\ \vdots & \vdots & \dots & \vdots \\ p_{n,1} & p_{n,2} & \dots & p_n \end{pmatrix}_{n\times n}$

and $\Phi_{M_4}(w_2) = \begin{pmatrix} q_1 & q_{1,2} & \cdots & q_{1,n} \\ q_{2,1} & q_2 & \cdots & q_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n,1} & q_{n,2} & \cdots & q_n \end{pmatrix}_{n\times n}$ satisfy the conditions $p_i = s \cdot q_i$, $(i = 1, 2, 3, \ldots, n)$, where s, $(s > 0)$ is a constant.

4.7.3 Searching M-ambiguous words using Ratio property:

In this subsection the relationship of ratio property and M-ambiguity is discussed. For simplicity tertiary sequences are used.

Ratio property of words over tertiary alphabet: Two words $w_1 < w_2$ over $\Sigma = \{a < b < c < d\}$ are said to satisfy the ratio property, written $w_1 \backsim_r w_2$ if

$$\Psi_{M_4}(w_1) = \begin{pmatrix} 1 & p_1 & p_{1,2} & p_{1,3} & p_{1,4} \\ 0 & 1 & p_2 & p_{2,3} & p_{2,4} \\ 0 & 0 & 1 & p_3 & p_{3,4} \\ 0 & 0 & 0 & 1 & p_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \Psi_{M_4}(w_2) = \begin{pmatrix} 1 & q_1 & q_{1,2} & q_{1,3} & q_{1,4} \\ 0 & 1 & q_2 & q_{2,3} & q_{2,4} \\ 0 & 0 & 1 & q_3 & q_{3,4} \\ 0 & 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

satisfy the conditions $p_i = s \cdot q_i$, $(i = 1, 2, 3, 4)$, $p_{i,i+1} = s \cdot q_{i,i+1}$, $(i = 1, 2, 3)$, $p_{i,i+2} = s \cdot q_{i,i+2}$, $(i = 1, 2)$ where s is a constant. The ratio property gives a sufficient condition for equality of the Parikh matrices. Let w_1, w_2, w_3 be three tertiary words such that the words satisfy the ratio property. One can form some words from w_1, w_2, w_3 as follows:

$w^i = w_1 w_2$, $w^{ii} = w_2 w_1$, $w^{iii} = {w_1}^n {w_2}^n$, $w^{iv} = (w_1 w_2)^n$;

$w^v = {w_1}^n {w_2}^n {w_3}^n$, $w^{vi} = (w_1 w_2 w_3)^n$, $w^{vi} = x^n {w_3}^n$, $w^{viii} = (x w_3)^n$.

It can be proved that the words formed by w_1, w_2, w_3 connecting them by the above way taking them as sub words give M-ambiguous words. Concatenation of two words satisfying the ratio property by making them subword of a word w makes w an M-ambiguous word. Thus the ratio property helps in finding certain M-ambiguous words out of certain sets of words satisfying ratio property. For $\Sigma = \{a < b < c < d\}$, the following lemmas from [87] are used for finding M-ambiguous words.

Lemma 1: For any three words w_1, w_2, w_3 over $\Sigma = \{a < b < c < d\}$ having the ratio property so that $w_1 \backsim_r w_2$, $w_2 \backsim_r w_3$ we have for $n \geq 0$ the following three properties,

1. $\Psi_{M_4}(w_1 w_2) = \Psi_{M_4}(w_2 w_1)$ i.e. w^i and w^{ii} are M-ambiguous.

2. $\Psi_{M_4}({w_1}^n {w_2}^n) = \Psi_{M_4}((w_1 w_2)^n)$ i.e. w^{iii} and w^{iv} are M-ambiguous.

3. $\Psi_{M_4}({w_1}^n {w_2}^n {w_3}^n) = \Psi_{M_4}((w_1 w_2 w_3)^n)$ i.e. w^{iii} and w^{iv} are M-ambiguous.

1. Proof:

If $\Psi_{M_4}(w_1) = \begin{pmatrix} 1 & p_1 & p_{1,2} & p_{1,3} & p_{1,4} \\ 0 & 1 & p_2 & p_{2,3} & p_{2,4} \\ 0 & 0 & 1 & p_3 & p_{3,4} \\ 0 & 0 & 0 & 1 & p_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and

$\Psi_{M_4}(w_2) = \begin{pmatrix} 1 & q_1 & q_{1,2} & q_{1,3} & q_{1,4} \\ 0 & 1 & q_2 & q_{2,3} & q_{2,4} \\ 0 & 0 & 1 & q_3 & q_{3,4} \\ 0 & 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ then

$\Psi_{M_4}(w_1 w_2) =$

$$\begin{pmatrix} 1 & p_1+q_1 & p_{1,2}+p_1q_2 & p_{1,3}+p_{1,2}q_3 & p_{1,4}+p_{1,3}q_4 \\ & & +q_{1,2} & +p_1q_{2,3}+q_{1,3} & +p_{1,2}q_{3,4}+p_1q_{2,4}+q_{1,4} \\ 0 & 1 & p_2+q_2 & p_{2,3}+p_2q_3+q_{2,3} & p_{2,4}+p_{2,3}q_4+p_2q_{3,4}+q_{2,4} \\ 0 & 0 & 1 & p_3+q_3 & p_{3,4}+p_3q_4+q_{3,4} \\ 0 & 0 & 0 & 1 & p_4+q_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$\Psi_{M_4}(w_2 w_1) =$

$$\begin{pmatrix} 1 & p_1+q_1 & p_{1,2}+p_2q_1 & p_{1,3}+p_{2,3}q_1+q_{1,3} & p_{1,4}+p_{2,4}q_1 \\ & & +q_{1,2} & +p_3q_{1,2} & +p_{3,4}q_{1,2}+p_4q_{1,3}+q_{1,4} \\ 0 & 1 & p_2+q_2 & p_{2,3}+p_3q_2+q_{2,3} & p_{2,4}+p_{3,4}q_2+p_4q_{2,3}+q_{2,4} \\ 0 & 0 & 1 & p_3+q_3 & p_{3,4}+p_4q_3+q_{3,4} \\ 0 & 0 & 0 & 1 & p_4+q_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

If the ratio property is satisfied for w_1, w_2 then we have

$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \frac{p_3}{q_3} = \frac{p_4}{q_4} = \frac{p_{1,2}}{q_{1,2}} = \frac{p_{1,3}}{q_{1,3}} = \frac{p_{2,3}}{q_{2,3}} = \frac{p_{2,4}}{q_{2,4}} = \frac{p_{3,4}}{q_{3,4}} = s$. These give

$p_1q_2 = p_2q_1, p_1q_{2,3} = p_{2,3}q_1, p_1q_{2,4} = p_{2,4}q_1,$

$p_2q_3 = p_3q_2, p_2q_{3,4} = p_{3,4}q_2,$

$p_3q_4 = p_4q_3, p_3q_{1,2} = p_{1,2}q_3$

$p_4q_{2,3} = p_{2,3}q_4,$

$p_{1,2}q_{3,4} = p_{3,4}q_{1,2}$. And thus $\Psi_{M_4}(w_1w_2) = \Psi_{M_4}(w_2w_1)$. That proves that w^i and w^{ii} are M-ambiguous.

2. Proof: $\Psi_{M_4}({w_1}^n{w_2}^n) = \Psi_{M_4}({w_1}^{n-1}w_1w_2{w_2}^{n-1})$

   $= \Psi_{M_4}({w_1}^{n-1})\Psi_{M_4}(w_1w_2)\Psi_{M_4}({w_2}^{n-1})$

   $= \Psi_{M_4}({w_1}^{n-1})\Psi_{M_4}(w_2w_1)\Psi_{M_4}({w_2}^{n-1})$ Using property (a) of lemma 1.

   $= \Psi_{M_4}({w_1}^{n-1})(w_2w_1)({w_2}^{n-1})$

   Repeatedly using property (a) we obtain $\Psi_{M_4}({w_1}^n{w_2}^n) = \Psi_{M_4}((w_1w_2)^n)$. That proves that w^{iii} and w^{iv} are M-ambiguous.

3. The proof is similar to the proof of (b) of lemma 1. Noting that $w_1 \backsim_r w_3$ which proves w^v and w^{vi} are M-ambiguous.

Lemma 2: Let w_1, w_2, w_3 be three nonempty words with $w_1 \backsim_r w_2$, $w_2 \backsim_r w_3$ and $x = w_1w_2$ then the following two properties are satisfied.

1. $\Psi_{M_4}(x^n{w_3}^n) = \Psi_{M_4}((xw_3)^n)$ i.e. w^{vii} and w^{viii} are M-ambiguous.

2. x is not related to w_3 under $\backsim_r$.

The proof of property (a) of lemma 2 is obvious from the lemma 1.i.e.

$\Psi_{M_4}(x^n {w_3}^n) = \Psi_{M_4}((xw_3)^n)$,

from which we get w^{vii} and w^{viii} are M-ambiguous.

The following example satisfies the property (b) of lemma 2.

Let $\Psi_{M_4}(w_1) = \begin{pmatrix} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$,

$\Psi_{M_4}(w_2) = \begin{pmatrix} 1 & 6 & 6 & 3 & 2 \\ 0 & 1 & 3 & 3 & 3 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$,

$\Psi_{M_4}(w_3) = \begin{pmatrix} 1 & 12 & 12 & 6 & 5 \\ 0 & 1 & 6 & 6 & 6 \\ 0 & 0 & 1 & 6 & 6 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

be three nonempty words. Here $w_1 \backsim_r w_2, w_2 \backsim_r w_3$.

$x = w_1 w_2$

$= \begin{pmatrix} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 6 & 6 & 3 & 2 \\ 0 & 1 & 3 & 3 & 3 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 8 & 14 & 16 & 18 \\ 0 & 1 & 4 & 7 & 10 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The above matrix is not related to w_3. It is an example that x is not related to w_3 under $\backsim_r$.

Ratio property is a property of words introduced in [87] helps in connecting words in an interesting way. The words satisfying ratio property can make certain M-ambiguous words. M-ambiguity of words is the problem for which Parikh matrices cannot be used in the injective way. That is for a single Parikh matrix satisfying M-ambiguity there may be more than one word corresponding to the matrix. This section presents generalization of ratio property and weak ratio property. Some lemmas proved for ternary sequences are investigated for tertiary words. Using those lemmas certain M-ambiguous words are formed from words satisfying ratio property by concatenating them in certain ways.

4.8 Solving M-ambiguity of words using M-ambiguity Reduction Factor:

4.8.1 Representation of M- ambiguous words by Parikh matrix together with M-ambiguity Reduction Factor:

For any word ζ in ternary sequence one can represent each word as repetition of $a^{x_i}b^{y_i}c^{z_i}$ where x_i = number of a s placed together in the word without being intercepted by b s and c s. y_i = number of b s placed together without being intercepted by c s and a s, z_i = number of c s placed together without being intercepted by a s and b s. So, the representation of a word ζ is $\zeta = a^{x_1}b^{y_1}c^{z_1}a^{x_2}b^{y_2}c^{z_2}\cdots a^{x_i}b^{y_i}c^{z_i}$. The value of i in this type of representation is determined by breaking of the sequence a s $<$ b s $<$ c s $<$ in the word. For example, let us take a ternary word $aabbbacc$. By the above process this word can be written as $a^2b^3c^0a^1b^0c^2$. After representing the word by above method the value of the function

$R(\zeta) = \sqrt{x_1{}^2 + y_1{}^2 + z_1{}^2} \cdot \sqrt{x_2{}^2 + y_2{}^2 + z_2{}^2} \cdots \sqrt{x_{l+m+n}{}^2 + y_{l+m+n}{}^2 + z_{l+m+n}{}^2}$

is calculated. The value of $R(\zeta)$ for the above word $aabbbacc$ is as follows:

$R(aabbbacc) = R(a^2b^3c^0a^1b^0c^2) = \sqrt{x_1{}^2 + y_3{}^2 + z_1{}^2} \cdot \sqrt{x_2{}^2 + y_2{}^2 + z_2{}^2}$

$= \sqrt{2^2 + 3^2 + 0}\sqrt{1^2 + 0 + 2^2} = \sqrt{4+9}\sqrt{1+4} = \sqrt{13}\sqrt{5}$. This calculated value $R(\zeta)$ is not always the same for the M-ambiguous words corresponding to a Parikh matrix. Some M-ambiguous words can have the same value of the function $R(\zeta)$ but some of them can have a unique value. So those words for which $R(\zeta)$ take a unique value can be sorted out and can be uniquely represented by the Parikh matrix and the unique value of $R(\zeta)$. This function $R(\zeta)$ is named as M-ambiguity reduction factor . Along with the Parikh matrix if we use this number for every word then the problem of M-ambiguity can be handled to a great extent.

4.8.2 Example of M-ambiguity Reduction factor for Binary sequence:

Let us consider the M-ambiguous words for the matrix over $\Sigma = \{a < b\}$ and the Parikh matrix be $\Psi_{M_2}(\xi) = \begin{pmatrix} 1 & 5 & 9 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$, the set of amiable words having this Parikh matrix is

$C = \{bbaababaaba, bbabaabaab, babbaababa, babbabaaab, bababbaaba,$

$bababbbaa, abbbabaaba, abbabbabaa, bbbaaaabab, bbaabbaaab,$

$bbabaaabba, baabbbabaa, abbbbaaaab, abbbaabbaa, ababbbbaaa, bbaaabbbaa\}$.

Let us find the respective numbers corresponding to the seven M-ambiguous words one by one.

$\xi_1 = bbaababaaba \to a^0b^2a^2b^1a^1b^1a^1b^1a^1b^0$

$\Rightarrow R(\xi_1) = \sqrt{0+2^2}\sqrt{2^2+1}\sqrt{1+1}\sqrt{1+1}\sqrt{1+0} = 4\sqrt{5}$

$\xi_2 = bbabaabaab \to a^0b^2a^1b^1a^2b^1a^2b^1 \Rightarrow R(\xi_2) = \sqrt{0+2^2}\sqrt{1+1}\sqrt{2^2+1}\sqrt{2^2+1} = 10\sqrt{2}$

$\xi_3 = babbaababa \to a^0b^1a^1b^2a^2b^1a^1b^1a^1b^0$

$\Rightarrow R(\xi_3) = \sqrt{0+1}\sqrt{1+2^2}\sqrt{2^2+1}\sqrt{1+1}\sqrt{1+0} = 5\sqrt{2}$

$\xi_4 = babbabaaab \to a^0b^1a^1b^2a^1b^1a^3b^1 \Rightarrow R(\xi_4) = \sqrt{0+1}\sqrt{1+2^2}\sqrt{1+1}\sqrt{3^2+1} = 10$

$\xi_5 = bababbaaba \to a^0b^1a^1b^1a^1b^2a^2b^1a^1b^0$

$\Rightarrow R(\xi_5) = \sqrt{0+1}\sqrt{1+1}\sqrt{1+2^2}\sqrt{2^2+1}\sqrt{1+0} = 5\sqrt{2}$

$\xi_6 = babababbaa \to a^0b^1a^1b^1a^1b^1a^1b^2a^2b^0$

$\Rightarrow R(\xi_6) = \sqrt{0+1}\sqrt{1+1}\sqrt{1+1}\sqrt{1+2^2}\sqrt{2^2+0} = 4\sqrt{5}$

$\xi_7 = abbbabaaba \to a^1b^3a^1b^1a^2b^1a^1b^0 \Rightarrow R(\xi_7) = \sqrt{1+3^2}\sqrt{1+1}\sqrt{2^2+1}\sqrt{1+0} = 10$

$\xi_8 = abbabbabaa \to a^1b^2a^1b^2a^1b^1a^2b^0 \Rightarrow R(\xi_8) = \sqrt{1+2^2}\sqrt{1+2^2}\sqrt{1+1}\sqrt{2^2+0} = 10\sqrt{2}$

$\xi_9 = bbbaaaabab \to a^0b^3a^4b^1a^1b^1 \Rightarrow R(\xi_9) = \sqrt{0+3^2}\sqrt{4^2+1}\sqrt{1+1} = 3\sqrt{34}$

$\xi_{10} = bbaabbaaab \to a^0b^2a^2b^2a^3b^1 \Rightarrow R(\xi_{10}) = \sqrt{0+2^2}\sqrt{2^2+2^2}\sqrt{3^2+1} = 8\sqrt{5}$

$\xi_{11} = bbabaaabba \to a^0b^2a^1b^1a^3b^2 \Rightarrow R(\xi_{11}) = \sqrt{0+2^2}\sqrt{1+1}\sqrt{3^2+2^2}\sqrt{1+0} = 2\sqrt{26}$

$\xi_{12} = baabbbabaa \to a^0b^1a^2b^3a^1b^1a^2b^0$

$\Rightarrow R(\xi_{12}) = \sqrt{0+1}\sqrt{2^2+3^2}\sqrt{1+1}\sqrt{2^2+0} = 2\sqrt{26}$

$\xi_{13} = abbbbaaaab \to a^1b^4a^4b^1 \Rightarrow R(\xi_{13}) = \sqrt{1+4^2}\sqrt{4^2+1} = 17$

$\xi_{14} = abbbaabbaa \to a^1b^3a^2b^2a^2b^0 \Rightarrow R(\xi_{14}) = \sqrt{1+3^2}\sqrt{2^2+2^2}\sqrt{2^2+0} =$

$8\sqrt{5}$

$\xi_{15} = ababbbbaaa \rightarrow a^1b^1a^1b^4a^3b^0 \Rightarrow R(\xi_{15}) = \sqrt{1+1}\sqrt{1+4^2}\sqrt{3^2+0} = 6\sqrt{5}$

$\xi_{16} = bbaaabbbaa \rightarrow a^0b^2a^3b^3a^2b^0 \Rightarrow R(\xi_{16}) = \sqrt{0+2^2}\sqrt{3^2+3^2}\sqrt{2^2+0} = 12\sqrt{2}$

Among these sixteen M-ambiguous words six words namely $\xi_5, \xi_7, \xi_9, \xi_{13}, \xi_{15}, \xi_{16}$ can be represented uniquely by Parikh matrix along with the reduction factor $R(\xi_i)$.

4.8.3 Examples of M-ambiguity Reduction factor for Ternary sequence:

1. Let us consider the M-ambiguous words for the following matrix. Let $\Sigma = \{a < b < c\}$ and the Parikh matrix be $\Psi_{M_3}(\zeta) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the set of amiable words having this Parikh matrix is $C = \{cbbac, cbbca, bccba\}$ Let us find the respective numbers corresponding to the three M-ambiguous words one by one.

   $\zeta_1 = cbbac \rightarrow a^0b^0c^1a^0b^2c^0a^1b^0c^1$

   $\Rightarrow R(\zeta_1) = \sqrt{0+0+1}\sqrt{0+2^2+0}\sqrt{1+0+1} = 2\sqrt{2}$

   $\zeta_2 = cbbca \rightarrow a^0b^0c^1a^0b^2c^1a^1b^0c^0$

   $\Rightarrow R(\zeta_2) = \sqrt{0+0+1}\sqrt{0+2^2+0}\sqrt{1+0+0} = 2$

   $\zeta_3 = bccba \rightarrow a^0b^1c^2a^0b^1c^0a^1b^0c^0$

   $\Rightarrow R(\zeta_3) = \sqrt{0+1+2^2}\sqrt{0+1+0}\sqrt{1+0+0} = \sqrt{5}$

   All these M- ambiguous words can be represented uniquely by Parikh matrix along with the reduction factor $R(\zeta_i)$.

2. Let us consider the M-ambiguous words for the matrix

$$\Psi_{M_3}(\alpha) = \begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

There are seven M-ambiguous words corresponding to the above matrix and these are $\alpha_1 = abbaccb$,$\alpha_2 = abbcacb$,$\alpha_3 = abbccab$,$\alpha_4 = baabccb$,$\alpha_5 = baacbbc$,$\alpha_6 = bacabbc$,$\alpha_7 = bcaabbc$.

Let us find the respective numbers corresponding to the seven M-ambiguous words one by one.

$\alpha_1 = abbaccb \to a^1b^2c^0a^1b^0c^2a^0b^1c^0$

$\Rightarrow R(\alpha_1) = \sqrt{1+2^2+0}\sqrt{1+0+2^2}\sqrt{0+1+0} = 2\sqrt{5}$

$\alpha_2 = abbcacb \to a^1b^2c^1a^1b^0c^1a^0b^1c^0$

$\Rightarrow R(\alpha_2) = \sqrt{1+2^2+1}\sqrt{1+0+1}\sqrt{0+1+0} = 2\sqrt{3}$

$\alpha_3 = abbccab \to a^1b^2c^2a^1b^1c^0$

$\Rightarrow R(\alpha_3) = \sqrt{1+2^2+2^2}\sqrt{1+1+0} = 3\sqrt{2}$

$\alpha_4 = baabccb \to a^0b^1c^0a^1b^0c^0a^1b^1c^2a^0b^1c^0$

$\Rightarrow R(\alpha_4) = \sqrt{0+1+0}\sqrt{1+0+0}\sqrt{1+1+2^2}\sqrt{0+1+0} = \sqrt{6}$

$\alpha_5 = baacbbc \to a^0b^1c^0a^1b^0c^0a^1b^0c^1a^0b^2c^1$

$\Rightarrow R(\alpha_5) = \sqrt{0+1+0}\sqrt{1+0+0}\sqrt{1+0+1}\sqrt{0+2^2+1} = \sqrt{2}\sqrt{5}$

$\alpha_6 = bacabbc \to a^0b^1c^0a^1b^0c^1a^1b^2c^1$

$\Rightarrow R(\alpha_6) = \sqrt{0+1+0}\sqrt{1+0+1}\sqrt{1+2^2+1} = 2\sqrt{3}$

$\alpha_7 = bcaabbc \to a^0b^1c^1a^1b^0c^0a^1b^2c^1$

$\Rightarrow R(\alpha_7) = \sqrt{0+1+1}\sqrt{1+0+0}\sqrt{1+2^2+1} = 2\sqrt{3}$

Among these seven M-ambiguous words four words namely $\alpha_1, \alpha_3, \alpha_4, \alpha_5$ can be represented uniquely by Parikh matrix along with the reduction factor $R(\alpha_i)$.

4.8.4 Example of M-ambiguity Reduction factor for Tertiary sequence:

Let us consider the tertiary sequences over $\Sigma = \{a < b < c < d\}$. The following two words $\beta_1 = abcddcbadcba, \beta_2 = abdccbddacba$ are M- ambiguous words corresponding to the Parikh matrix $\Psi_{M_4}(\beta) = \begin{pmatrix} 1 & 3 & 4 & 4 & 4 \\ 0 & 1 & 3 & 4 & 4 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Let us find the respective numbers corresponding to the two M-ambiguous words one by one.

$\beta_1 = abcddcbadcba \rightarrow a^1b^1c^1d^2a^0b^0c^1d^0a^0b^1c^0d^0a^1b^0c^0d^1a^0b^0c^1d^0a^0b^1c^0d^0a^1b^0c^0d^0$

$\Rightarrow R(\beta_1) = \sqrt{1+1+1+2^2}\sqrt{0+0+1+0}\sqrt{0+1+0+0}\sqrt{1+0+0+1}$
$\sqrt{0+0+1+0}\sqrt{0+1+0+0}\sqrt{1+0+0+0} = \sqrt{14}$

$\beta_2 = abdccbddacba \rightarrow a^1b^1c^0d^1a^0b^0c^2d^0a^0b^1c^0d^2a^1b^0c^1d^0a^0b^1c^0d^0a^1b^0c^0d^0$

$\Rightarrow R(\beta_2) = \sqrt{1+1+0+1}\sqrt{0+0+2^2+0}\sqrt{0+1+0+2^2}\sqrt{1+0+1+0}$
$\sqrt{0+1+0+0}\sqrt{1+0+0+0} = 4\sqrt{5}$

All these M- ambiguous words can be represented uniquely by Parikh matrix along with the reduction factor $R(\beta_i)$.

4.9 Conclusion of the Chapter

In this chapter two types of algorithm are developed. One is meant for finding Parikh matrix corresponding to a word and other is for finding the words from the matrix. The first one is extended from binary to ternary and tertiary alphabets. The second one is extended from binary to ternary alphabet. In the third section the graphical representation of Formal words are given. The notion is extended from binary to ternary alphabet. In the next section a set of equations to find formal words corresponding to a Parikh matrix is developed. The set is extended from binary to ternary and tertiary alphabets. Comparing M-ambiguous words by Stepping distance is introduced in the fifth section.The notion is extended from binary to ternary and tertiary alphabets. Theorems regarding Parikh matrices are given in the next section. In the seventh section some observations about Ratio property are given. In the eighth section M-ambiguity Reduction factor is introduced.

*****