Chapter 3

Observational technique: Polarimetric

In this chapter, polarimetric technique (optical) to observe astronomical objects has been discussed. It begins with a note on early astronomical polarimetric observation and proceeds to mention some sources of astrophysical polarization. Finally, astronomical polarimetry in optical band has been described with derivation of theoretical formulae for calculation.

3.1 Polarimetry

It is the measurement and interpretation of the polarized electromagnetic waves (x-ray, radio, light waves). All astronomical sources are polarized to some degree. Polarimetry plays a key role to study those objects and is therefore a powerful astronomical technique to be employed. It has accountably contributed in the development of modern astronomy, providing insight into physical processes occurring in systems that range from our own solar system to high-redshift galaxies. Sunlight scattered by a cometary atmosphere becomes partially polarized and polarimetry on them gives information on the nature, shape, structure and sizes of the cometary particles. It also complements photometric, spectroscopic analyses of sources of radiation and has made many astrophysical discoveries.

3.2 Early observations

As mentioned in the previous chapter, the study of polarized light began with the discovery of birefringence in crystals by Erasmus Bartholinus and its subsequent interpretation by Christian Huygens around the year 1670 [Brosseau, 1998]. Astronomical observations of polarized light commenced in the middle of the 19th century. Some of the early publications have discussed the non-zero polarization in comet ([Arago, 1854], [Arago, 1857]), the linear polarization of Sun light reflected by the Moon [Secchi, 1860] and the linear polarization of the light from the solar corona [Edlund, 1860]. Subsequently, the field of polarimetry evolved closely with the technical progress of observations in general: from optical polarimetry to radio polarimetry in the 1940s [Wilson et al., 2009] and eventually to space-based X-ray polarimetry in the 1970s [Weisskopf et al., 1978].

3.3 Sources of astrophysical polarization

Polarized radiation is a widespread phenomenon. However, it is sometimes hard to notice because the polarization is averaged down to a level below the detection limit for observations. Therefore astronomical polarimetry has “only” a few, but important applications where the polarization signal is strong and provides interesting information about certain objects; some of which are discussed below-

3.3.1 Light scattering:

This occurs due to the fact that the EM waves cause the electron in media to vibrate, producing radiation and causing polarization of the light. Scattering processes occur everywhere in astronomy. Phenomena with strong scattering polarization are:

i. Scattered solar light: scattered sunlight from planets, moons, asteroids, comets and other solar system objects is polarized partially. ([Sen et al., 1990], [Hadamcik and Levasseur-Regourd, 2003b])

ii. Circumstellar scattering by dust and gas of stars undergoing mass loss or stars in formation produces partial polarization. [Halonen et al., 2013]

iii. Circumnuclear scattering in AGN (Active Galactic Nuclei) produces strong linear polarization due to the material near the accreting super-massive black hole. [Yan and Lazarian, 2007]

3.3.2 Zeeman effect:

The Zeeman effect describes the line emission and absorption of atoms and molecules in magnetized materials (e.g. plasmas). Some of the phenomena are-

i. Magnetized solar atmosphere, where the small scale structure of the magnetic fields (e.g. solar spots) can be investigated in detail. ([Berdyugina et al., 2003], [Stenflo, 2013])

ii. Magnetized stars, with strong and large scale magnetic fields, like a global dipole field, or a stellar spot which dominates the surface. [Ho and Lai, 2003]

iii. Interstellar magnetic fields can be measured from the polarization of emission lines. [Beck, 1986]

3.3.3 Synchrotron emission:

Synchrotron and cyclotron radiation by fast moving particles (mostly electrons) in magnetic field produce linear and circular emission.

i. Stellar corona produce relativistic electrons triggered by the magnetic activity of the star and they emit strongly variable, polarized synchrotron radio emission. [Dulk, 1985]

ii. Relativistic electrons in the interstellar and intergalactic medium emit highly polarized circular and linear polarization in the radio range. ([Gardner and Whiteoak, 1966], [Lequeux, 2004])

iii. Relativistic jets from quasars and gamma ray bursts produce linear and circular polarization. [Lyutikov et al., 2003]

3.3.4 Dichroic absorption:

The absorption of light is polarization dependent if the orientation of anisotropic absorbers has a predominant direction due to the alignment by magnetic fields in astrophysical cases.

i. interstellar dust is magnetically aligned by galactic magnetic fields and they can produce a linear polarization. ([Crutcher et al., 2003], [Ruzmaikin et al., 2013])

3.3.5 Birefringence:

Birefringence is the polarization dependent refraction by a medium because of a predominant orientation of the medium structure. For plasma gas a magnetic field with a predominant direction can produce a circular birefringence (Faraday rotation).

i. Interstellar Faraday rotation: the interstellar medium has a circular birefringence proportional to the product of the line of sight magnetic field and the electron column density. This product is the so-called rotation measure, which defines the rotation of the linear polarization of background radio sources. [Guo-jun et al., 1995]

3.4 Astronomical polarimetry (optical)

Similar to photometry and spectroscopy, the techniques for polarimetry of radiation from astronomical sources depend strongly upon the energy of the light. The techniques can be distinguished according to wavelength regimes (Radio, Optical, X-ray). At optical wavelengths, polarimetry is limited to light intensities rather than electric waves. Thus basic aim of optical polarimetry is to measure the light flux from a celestial object at several wavelengths. Along with optical filters, polarimetric measurements require the use of some auxiliary optical elements placed in the optical path before the detector (usually a CCD array). Additional auxiliary components are mainly polarizers, polarization beam splitters, and retarder plates.

Linear polarization can be calculated by measuring the polarization parameters i.e. Stokes parameters (introduced in Chapter 2). Insertion of above mentioned auxiliary elements changes the polarization state of incoming beam. Hans Müller developed a formalism to treat these interactions; this will be discussed in next section (section 3.4).

There are two ways to measure the polarization:

Single beam polarimetry: measuring the two or more polarization modes consecutively with an instrument that can measure different polarization states.

Double beam polarimetry: measuring the two polarization modes simultaneously with an instrument which can split the opposite polarization modes.

Single and double beam polarimetry will be discussed in correlation with the two telescopes used for polarimetric observations in the present work. First one is the IUCAA (Inter University Center for Astronomy and Astrophysics) Girawali Observatory, "IGO" ; situated in Pune, India. Second one is Haute-Provence Observatory, "OHP" , situated near Marseille, France.

3.4.1 Single beam polarimetry:

A single beam polarimetry performs measurements in two or more alternating polarization modes consecutively. An exposure from a single-beam polarimeter consists of an image of the target in a single state of polarization. Some form of analyser within the polarimeter removes all but the required state of polarization from the incoming light, which then goes on to be recorded on a suitable imaging device (usually CCD camera). The exposures are continued for different polarizing modes consecutively.

The polarimeter: There are several varieties of single-beam polarimeter depending upon arrangements of analyser.-

(1) A single analyser which passes only light polarized parallel to a specified axis. The analyser is rotated to measure light polarized in different directions.

(2) Several fixed analysers, each of which passes only light polarized parallel to its axis. The required analyser is inserted into the light path to measure light polarized in different directions. In case of OHP, four polaroid filters are mounted on a rotating wheel. ([Hadamcik et al., 2010], [Hadamcik et al., 2013])

(3) A single fixed analyser which passes light polarized parallel to its axis. A half-wave plate is placed in front of the analyser. The plane of polarization of the incoming light is rotated by rotating the HWP before it reaches the analyser. Light polarized in different directions can thus be measured.

The Observational Procedure: There will be a reference direction within the focal plane. The orientation of the rotating analyser or half-wave plate is specified by giving the angle between the reference direction and the analyser or half-wave plate. At least three exposures with different analyser or half-wave plate positions are required to estimate the degree and orientation of the polarization [Sen et al., 1990]. For each orientation of analyser or HWP, a polarized intensity image is recorded. For OHP, the fast axis of four polaroid filters, mounted on a rotating wheel, are oriented at $45°$ from one another, the first one corresponding to the so-called direction Zero “0” . The corresponding recorded intensities being I_0, I_{45}, I_{90} and I_{135}. Exposures at four analyser positions are always better to reduce the noise. ([Hadamcik et al., 2010], [Hadamcik et al., 2013])

3.4.2 Double beam polarimetry:

A double beam polarimetry is performing measurements in two polarization modes simultaneously. An exposure from a double-beam polarimeter consists of an image of the target in two states of polarization. The polarimeter consists of some analyser which can split light into the two polarization modes, one perpendicular to other. The advantage of this system over a single beam instrument is that variations in sky background between exposures affect both states of polarization equally, and so can be eliminated.

The Polarimeter: A dualbeam polarimeter suitable for measuring linear polarization usually contains the following optical components- i)A focalplane mask, ii)A halfwave plate, iii)An analyser, iv)A detector. ([Sen and Tandon, 1994], [Ramaprakash et al., 1998])

The light collected by the telescope passes through these components in the order listed (see Fig. (3.1)).

The heart of the polarimeter is the analyser (Wallaston prism in case of IGO), which splits incoming partially plane polarized light into two (ordinary and extraordinary) beams. Ordinary (O) ray contains the component of the incoming light which is polarized parallel to the axis of the analyser, and the extraordinary (E) ray contains that component which is polarized orthogonal to the axis of the analyser.

These two beams are recorded simultaneously on a suitable detector such as a CCD. On the detector, the two beams form two images displaced by some distance

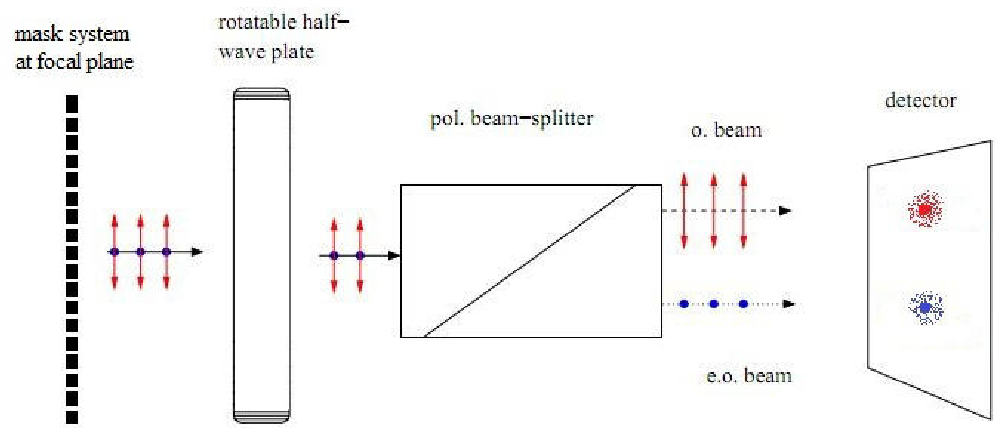

Figure 3.1: The main optical components in a typical dual-beam imaging polarimeter.

(0.9 arcmin for IGO) determined by the design of the instrument; both images representing the same area of the sky.

A masking system is used to prevent any overlap between the two images. In some instruments this takes the form of a series of parallel, equally spaced bars in the focal plane of the telescope (grid in the focal plane for IGO). There are several other systems such as a mask containing only a single aperture, but the principle is the same.

If the incoming light is only partially polarized, then at least two exposures are required to estimate both the degree and the orientation of the polarization. This is achieved by placing a halfwave plate in front of the analyser, and rotating it in steps of $22.5°$, resulting in a rotation of the plane of polarization of $45°$. Using this scheme the positions of the O and E ray images on the detector are unchanged.The orientation of the plane of polarization of the incoming light is measured relative to a fixed "reference" direction. The analyser axis and the $0°$ position of the halfwave plate are usually parallel to this direction.

The Observational Procedure: At least two exposures are required to estimate the degree and orientation of the polarization, taken with halfwave plate positions of $0°$ and $22.5°$ (the exposures being $I(0)$ and $I(22.5)$). The O and E ray images in $I(0)$ measure the intensities ($I_o(0)$ and $I_e(0)$) at angles of $0°$ and $90°$) to the reference direction. $I(22.5)$ has an effective analyser angle of $45°$ (twice the halfwave plate angle) and so measures the intensities ($I_o(22.5)$ and $I_e(22.5)$) at angles of $45°$ and $135°$ to the reference direction. Usually, a further two exposures are taken at halfwave plate positions of $45°$ and $67.5°$ (referred to here as exposures $I(45)$ and $I(67.5)$). These provide some redundancy in the data and enable internal consistency checks to be made during the data reduction stage.

3.5 The Müller matrices for polarizing components

When an optical beam interacts with matter its polarization state is almost always changed. The polarization state can be changed by-

(1) An optical element (polarizer or diattenuator) that changes the orthogonal amplitudes unequally.

(2) An optical device (retarder, wave plate, compensator, or phase shifter) that introduces a phase shift between the orthogonal components.

(3) An optical device (rotator) that rotates the orthogonal components of the beam through an angle as it propagates through the element.

(4) An optical device (depolarizer) that transforms energy in polarized states going to the unpolarized state.

Considering the transverse field components for a plane wave are-

$$E_x(z,t) = E_{0x} \cos(\omega t - kz + \delta_x) \tag{3.1a}$$

$$E_y(z,t) = E_{0y} \cos(\omega t - kz + \delta_y) \tag{3.1b}$$

Equation (3.1a, 3.1b) can be changed by varying the amplitudes, E_{0x} or E_{0y}, or the phase, δ_x or δ_y and, finally, the direction of $E_y(z,t)$ and $E_y(z,t)$. The corresponding devices for causing these changes are the polarizer, retarder, and rotator. These three are usually used for polarimetric devices designed for astronomical observation. Thus, interaction of polarized light with these three polarizing elements changes the polarization state of an optical beam. Hans Müller [Müller, 1948] extends and combines Jones calculus and Stokes formalism to treat such interactions.

Fig. (3.2) shows an incident beam (characterized by its Stokes parameters S_i)

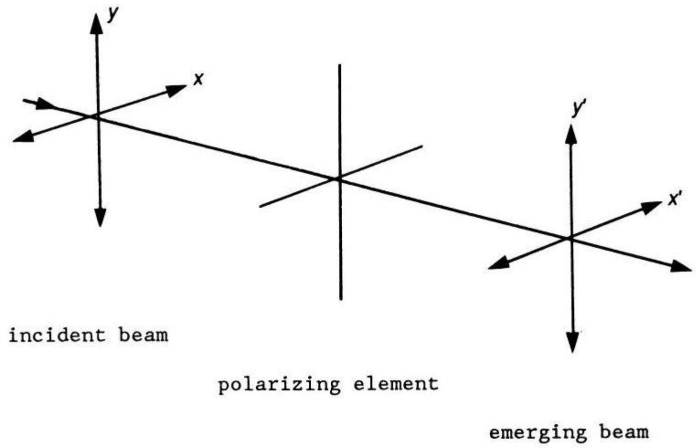

Figure 3.2: Interaction of a polarized beam with a polarizing element.

interacting with a polarizing element and the emerging beam (characterized by a new set of Stokes parameters S_i'); where i=0,1,2,3. We assume that S_i' can be expressed as a linear combination of the S_i as:

$$\begin{aligned}
S_0' &= m_{00}S_0 + m_{01}S_1 + m_{02}S_2 + m_{03}S_3 \\
S_1' &= m_{10}S_0 + m_{11}S_1 + m_{12}S_2 + m_{13}S_3 \\
S_2' &= m_{20}S_0 + m_{21}S_1 + m_{22}S_2 + m_{23}S_3 \\
S_3' &= m_{30}S_0 + m_{31}S_1 + m_{32}S_2 + m_{33}S_3
\end{aligned} \tag{3.2}$$

In matrix form

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{3.3}$$

$$S' = M.S \tag{3.4}$$

([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

3.5.1 Müller matrix for polarizer:

A polarizer (generator or analyzer) is an optical element that attenuates the orthogonal components of an optical beam unequally (anisotropic attenuator). If the orthogonal components of the incident beam are attenuated equally, then the polarizer becomes a neutral density filter.

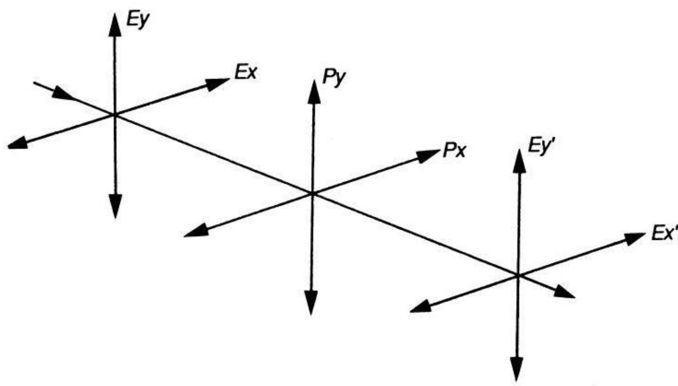

Figure 3.3: The Müller matrix of a polarizer with attenuation coefficients p_x and p_y.

In fig. (3.3), a polarized beam is shown incident on a polarizer along with the emerging beam. The components of the incident beam and emergent beam are represented by E_x, E_y and E'_x, E'_y and they are parallel to the original axes. The fields are related by

$$E'_x = p_x E_x \qquad 0 \leq p_x \leq 1 \tag{3.5a}$$

$$E'_y = p_y E_y \qquad 0 \leq p_y \leq 1 \tag{3.5b}$$

The factors p_x and p_y are the amplitude attenuation coefficients along orthogonal transmission axes. For no attenuation (perfect transmission) along an orthogonal axis $p_x p_y$=1, and for complete attenuation $p_x p_y$=0. If one of the axes has a zero absorption coefficient (there is no transmission along this axis), the polarizer is said to have only a single transmission axis.

The Stokes polarization parameters of the incident and emerging beams are respectively,

$$S_0 = E_x E_x^* + E_y E_y^* \tag{3.6a}$$
$$S_1 = E_x E_x^* - E_y E_y^* \tag{3.6b}$$
$$S_2 = E_x E_y^* + E_y E_x^* \tag{3.6c}$$
$$S_3 = i(E_x E_y^* - E_y E_x^*) \tag{3.6d}$$

and

$$S_0' = E_x' E_x'^* + E_y' E_y'^* \tag{3.7a}$$
$$S_1' = E_x' E_x'^* - E_y' E_y'^* \tag{3.7b}$$
$$S_2' = E_x' E_y'^* + E_y' E_x'^* \tag{3.7c}$$
$$S_3' = i(E_x' E_y'^* - E_y' E_x'^*) \tag{3.7d}$$

Substituting Eq. (3.5) into Eq. (3.7) and using Eq. (3.6),

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} p_x^2 + p_y^2 & p_x^2 - p_y^2 & 0 & 0 \\ p_x^2 - p_y^2 & p_x^2 + p_y^2 & 0 & 0 \\ 0 & & 2p_x p_y & 0 \\ 0 & & 0 & 2p_x p_y \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{3.8}$$

The 4×4 Müller matrix for a polarizer with amplitude attenuation coefficients p_x and p_y is written as

$$M = \frac{1}{2} \begin{pmatrix} p_x^2 + p_y^2 & p_x^2 - p_y^2 & 0 & 0 \\ p_x^2 - p_y^2 & p_x^2 + p_y^2 & 0 & 0 \\ 0 & & 2p_x p_y & 0 \\ 0 & & 0 & 2p_x p_y \end{pmatrix} \tag{3.9}$$

Finally, the Müller matrix of a general linear polarizer in terms of trigonometric functions is obtained by setting

$$p_x^2 + p_y^2 = p^2 \tag{3.10a}$$

$$p_x = p\cos\gamma \qquad p_y = p\sin\gamma \tag{3.10b}$$

Substituting Eq. (3.10) into Eq. (3.9) yields

$$M = \frac{p^2}{2}\begin{pmatrix} 1 & \cos 2\gamma & 0 & 0 \\ \cos 2\gamma & 1 & 0 & 0 \\ 0 & 0 & \sin 2\gamma & 0 \\ 0 & 0 & 0 & \sin 2\gamma \end{pmatrix} \tag{3.11}$$

where $0 \leq \gamma \leq 90°$. For an ideal perfect linear polarizer p=1. For a linear horizontal polarizer γ=0, and for a linear vertical polarizer $\gamma = 90°$.
([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

3.5.2 Müller matrix for retarder:

A retarder is a polarizing element which changes the phase of the optical beam. Other names being phase shifter, wave plate, and compensator. Retarders introduce a phase shift of ϕ between the orthogonal components of the incident field. This can be thought of as being accomplished by causing a phase shift of $\phi/2$ along the x axis and a phase shift of $-\phi/2$ along the y axis. These axes of the retarder are referred to as the fast and slow axes, respectively.

In fig. (3.4), the incident and emerging beam and the retarder **have** been shown. The components of the emerging beam are related to the incident beam by-

$$E'_x(z,t) = \exp(+\phi/2)E_x(z,t) \tag{3.12a}$$

$$E'_y(z,t) = \exp(-\phi/2)E_y(z,t) \tag{3.12b}$$

Substituting Eq. (3.6) into the equations defining the Stokes parameters Eq.

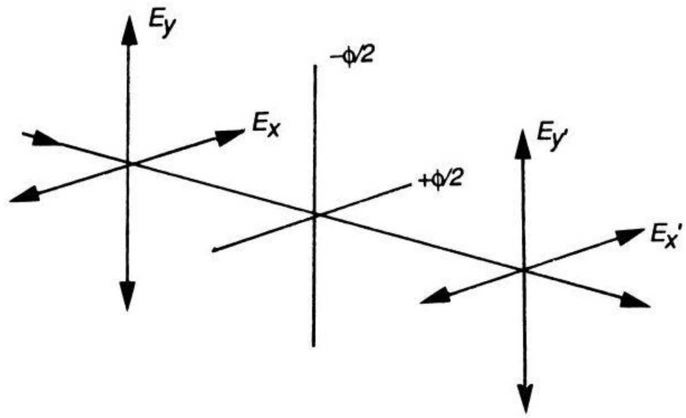

Figure 3.4: Propagation of a polarized beam through a retarder.

(3.6) and (3.7), we get

$$S_0' = S_0 \tag{3.13a}$$
$$S_1' = S_1 \tag{3.13b}$$
$$S_2' = S_2 \cos\phi + S_3 \sin\phi \tag{3.13c}$$
$$S_3' = -S_2 \sin\phi + S_3 \cos\phi \tag{3.13d}$$

In matrix form-

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{3.14}$$

The Mueller matrix for a retarder with a phase shift ϕ is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{3.15}$$

An important type of wave retarder is the half-wave retarder $\phi = 180°$. For this condition Eq. (3.15) reduces to

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.16}$$

Using the relation $S' = MS$

$$S' = \begin{pmatrix} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{pmatrix} = \begin{pmatrix} S_0 \\ S_1 \\ -S_2 \\ -S_3 \end{pmatrix} \tag{3.17}$$

Thus

$$\tan 2\psi' = \frac{S'_2}{S'_1} = \frac{-S_2}{S_1} = -\tan 2\psi \tag{3.18a}$$

$$\sin 2\chi' = \frac{S'_3}{S'_0} = \frac{-S_3}{S_0} = -\sin 2\chi \tag{3.18b}$$

and

$$\psi' = 90° - \psi \tag{3.19a}$$

$$\chi' = 90° - \chi \tag{3.19b}$$

Half-wave retarders possess the property that they can rotate the polarization ellipse.
([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

3.5.3 Müller matrix for rotator:

Another way to change the polarization state of an optical field is to allow a beam to propagate through a polarizing element that rotates the orthogonal field components $E_x(z,t)$ and $E_y(z,t)$ through an angle θ.

In order to derive the Müller matrix for rotation, we consider Fig. (3.5). The angle θ describes the rotation of E_x to E'_x and of E_y to E'_y. Similarly, the angle β is the angle between E and E_x.

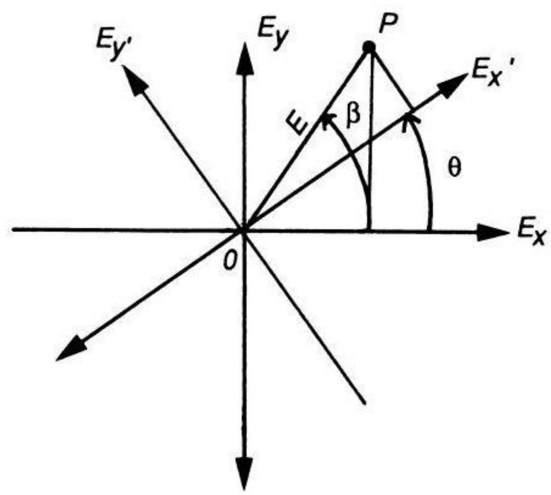

Figure 3.5: Rotation of the optical field components by a rotator.

In fig. (3.5), the point P is described in the E'_x, E'_y coordinate system by

$$E'_x = E_x \cos\theta + E_y \sin\theta \tag{3.20a}$$

$$E'_y = -E_x \sin\theta + E_y \cos\theta \tag{3.20b}$$

In the E_x, E_y coordinate system we have

$$E_x = E\cos\beta \tag{3.21a}$$

$$E_y = E\sin\beta \tag{3.21b}$$

Expanding trigonometric functions

$$E'_x = E(\cos\beta\cos\theta + \sin\beta\sin\theta) \tag{3.22a}$$

$$E'_y = E(\sin\beta\cos\theta - \sin\theta\cos\beta) \tag{3.22b}$$

The Müller matrix for rotation using Eq. (3.20):

$$M_R(2\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.23}$$

$$\tan 2\psi' = \frac{-S_1\sin 2\theta + S_2\cos 2\theta}{S_1\cos 2\theta + S_2\sin 2\theta} \tag{3.24}$$

$$S_2 = S_1\tan 2\psi \tag{3.25}$$

$$\tan 2\psi' = \tan(2\psi - 2\theta) \tag{3.26}$$

$$\psi' = \psi - \theta \tag{3.27}$$

Equation (3.27) shows that a rotator merely rotates the polarization ellipse of the incident beam; the ellipticity remains unchanged. The sign is negative in (3.27) because the rotation is clockwise.
([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

3.5.4 Müller matrix for rotated polarizing components:

Referring to Fig. (3.6), incoming S beam interacts with rotating polarizing component characterized by Müller matrix $M_R(2\theta)$. The emerging beam $S'(= M_R(2\theta)S)$ Interacts with polarizing element of Müller matrix M. The Stokes vector of the emerging beam is $S''(= MS')$. Finally, we must take the components of the emerging beam along the original x and y axes as shown in Fig. 5-6. This

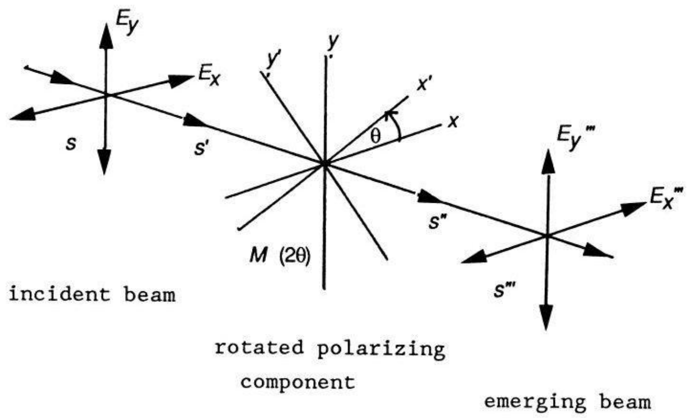

Figure 3.6: Derivation of the Müller matrix for rotated polarizing components.

can be described by a counterclockwise rotation of S'' through $-\theta$ and back to the original x, y axes. Thus the emerging beam $S'''(= M_R(-2\theta)S'')$ is

$$S''' = M(2\theta)S \tag{3.28}$$

Where

$$M(2\theta) = M_R(-2\theta)MM_R(2\theta) \tag{3.29}$$

Müller matrix for rotated polarizer: Carrying out the matrix multiplication according to Eq. (3.29) and using Eqns. (3.23), (3.11) and setting p^2 to unity:

$$M = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\gamma \cos 2\theta & \cos 2\gamma \sin 2\theta & 0 \\ \cos 2\gamma \cos 2\theta & \cos^2 2\theta + \sin 2\gamma \sin^2 2\theta & (1-\sin 2\gamma)\sin 2\theta \cos 2\theta & 0 \\ \cos 2\gamma \sin 2\theta & (1-\sin 2\gamma)\sin 2\theta \cos 2\theta & \sin^2 2\theta + \sin 2\gamma \cos^2 2\theta & 0 \\ 0 & 0 & 0 & \sin 2\gamma \end{pmatrix} \tag{3.30}$$

For an ideal linear horizontal polarizer ($\gamma = 0$). Eq. (3.30) reduces to

$$M_P(2\theta) = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & \sin 2\gamma \end{pmatrix} \tag{3.31}$$

Müller matrix for a retarder or wave plate: Carrying out the matrix multiplication according to Eq. (3.29) and using Eqns. (3.23), (3.15)-

$$M_c(\phi, 2\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta + \cos\phi \sin^2 2\theta & (1-\cos\phi)\sin 2\theta \cos 2\theta & -\sin\phi \sin 2\theta \\ 0 & (1-\cos\phi)\sin 2\theta \cos 2\theta & \sin^2 2\theta + \cos\phi \cos^2 2\theta & \sin\phi \cos 2\theta \\ 0 & \sin\phi \sin 2\theta & -\sin\phi \cos 2\theta & \cos\phi \end{pmatrix} \tag{3.32}$$

For a half-wave retarder, the phase shift is $\phi = 180°$. Eq. (3.32) reduces to

$$M_c(180°, 4\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4\theta & \sin 4\theta & 0 \\ 0 & \sin 4\theta & -\cos 4\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.33}$$

The Mueller matrix is an appropriate formalism for characterizing polarization measurements because it contains within its elements all of the polarization properties : diattenuation, retardance, depolarization, and their form, either linear, circular, or elliptical. When the Mueller matrix is known, then the exiting polarization state is known for an arbitrary incident polarization state.
([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

3.6 Calculating the polarization

Suppose that the y-component of electric field is subjected to a retardation ϕ with respect to x-component (using a compensator or HWP). Also let $I(\theta, \phi)$ be the intensity of light vibration in the direction which makes an angle θ with the positive x-direction. This intensity will be observed by sending light through a polarizer with appropriate orientation.

The component of electric vector in θ direction after retardation ϕ has been introduced is

$$E(t, \theta, \phi) = E_x \cos\theta + E_y e^{i\phi} \sin\theta \tag{3.34}$$

Thus,

$$\begin{aligned} I(\theta, \phi) &=< E(t, \theta, \phi) E^*(t, \theta, \phi) > \\ &=< (E_x \cos\theta + E_y e^{i\phi} \sin\theta)(E_x^* \cos\theta + E_y^* e^{-i\phi} \sin\theta) > \\ &= J_{xx} \cos^2\theta + J_{yy} \sin^2\theta + J_{xy} \cos\theta \sin\theta e^{-i\phi} + J_{yx} \sin\theta \cos\theta e^{i\phi} \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} J_{xx} &=< E_x E_x^* > \\ J_{yy} &=< E_y E_y^* > \\ J_{xy} &=< E_x E_y^* > \\ J_{yx} &=< E_y E_x^* > \end{aligned} \tag{3.36}$$

As discussed in sections 3.4.1 and 3.4.2, one needs to take exposures to record the intensity for several values of θ (orientation of polarizer) and ϕ (delay introduced by compensator). Let $\{\theta, \phi\}$ denotes the measurement corresponding to the pair θ, ϕ.

$$\begin{aligned} I(0°, 0) &= J_{xx} \\ I(90°, 0) &= J_{yy} \\ I(45°, 0) &= \frac{1}{2}(J_{xx} + J_{yy}) + \frac{1}{2}(J_{xy} + J_{yx}) \\ I(135°, 0) &= \frac{1}{2}(J_{xx} + J_{yy}) - \frac{1}{2}(J_{xy} + J_{yx}) \end{aligned}$$

$$I(45^\circ, \frac{\pi}{2}) = \frac{1}{2}(J_{xx} + J_{yy}) + i\frac{1}{2}(J_{yx} - J_{xy})$$
$$I(135^\circ, \frac{\pi}{2}) = \frac{1}{2}(J_{xx} + J_{yy}) - i\frac{1}{2}(J_{yx} - J_{xy}) \tag{3.37}$$

It follows from above eqn. (3.37)-

$$I(0^\circ, 0) + I(90^\circ, 0) = J_{xx} + J_{yy} =< E_x E_x^* > + < E_y E_y^* >= S_0$$
$$I(0^\circ, 0) - I(90^\circ, 0) = J_{xx} - J_{yy} =< E_x E_x^* > - < E_y E_y^* >= S_1$$
$$I(45^\circ, 0) - I(135^\circ, 0) = J_{xy} + J_{yx} =< E_x E_y^* > + < E_y E_x^* >= S_2$$
$$I(45^\circ, \frac{\pi}{2}) - I(135^\circ, \frac{\pi}{2}) = i(J_{yx} - J_{xy}) = i(< E_x E_y^* > - < E_y E_x^* >) = S_3 \tag{3.38}$$

Single beam polarimetry: To estimate the degree and orientation of the polarization, let the exposures recorded be I_0, I_{45}, I_{90} and I_{135} corresponding to $0^\circ, 45^\circ, 90^\circ$ and 135° of the polaroid. In case of single beam polarimetry, let us denote-

$$I(0^\circ, 0) = I_0; \quad I(90^\circ, 0) = I_{90}; \quad I(45^\circ, 0) = I_{45}; \quad I(135^\circ, 0) = I_{135}$$

Thus,

$$S_0 = I = I_0 + I_{90} = I_{45} + I_{135} \tag{3.39a}$$
$$S_1 = Q = I_0 - I_{90} \tag{3.39b}$$
$$S_2 = U = I_{45} - I_{135} \tag{3.39c}$$

Linear polarization (P), and polarization angle (θ) are calculated by the expressions-

$$P = \frac{\sqrt{Q^2 + U^2}}{I} \tag{3.40a}$$

$$\theta = 0.5 \arctan(U/Q) \tag{3.40b}$$

Double beam polarimetry: In this case, the ordinary and extra ordinary components corresponding to $0°, 22.5°, 45°$ and $67.5°$ position of HWP are-

$(I_o(0), I_e(0)), (I_o(22.5), I_e(22.5)), (I_o(45), I_e(45))$ and $(I_o(67.5), I_e(67.5))$

Thus

$$I(\beta) = \frac{\frac{I_e(\beta)}{I_o(\beta)} - 1}{\frac{I_e(\beta)}{I_o(\beta)} + 1} \qquad \beta = 0, 22.5, 45, 67.5 \tag{3.41a}$$

$$Q(\beta) = \frac{\frac{I_e(\beta)}{I_o(\beta)} - 1}{\frac{I_e(\beta)}{I_o(\beta)} + 1} \qquad \beta = 0, 45 \tag{3.41b}$$

$$U(\beta) = \frac{\frac{I_e(\beta)}{I_o(\beta)} - 1}{\frac{I_e(\beta)}{I_o(\beta)} + 1} \qquad \beta = 22.5, 67.5 \tag{3.41c}$$

Polarization (P) and polarization angle (θ) are given by-

$$P = \frac{\sqrt{Q^2 + U^2}}{I} = \sqrt{I(\beta)^2 + I(\beta + 22.5)^2} \tag{3.42a}$$

$$\theta = 0.5 \arctan(U/Q) = 0.5 \arctan(I(\beta + 22.5)/I(\beta)) \tag{3.42b}$$

([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

3.7 Polarimetric calibration measurements

In addition to these target exposures, some flat-field exposures are also required. These are used to correct for any spatial variation in the sensitivity of the system, and consist of exposures of a photometrically flat surface. It is important that the flatfield has a good signal-to-noise ratio. For this reason, it is better to take one or more flatfield exposures at each analyser position, and stack them together into a single master flat field.

Ideally, the flat field sources should be unpolarized. However, a spatially constant polarization across the flat field source can be corrected during data reduction with the help of some additional target exposures. Since the polarization of the flat field surface is rarely known to be zero, these additional target exposures should always be taken.

The sky polarization is defined by the N-S direction for the zero point for the orientation of the linear polarization and the polarization position angle θ is measured from N over E. Thus, the polarimetric-specific calibration steps include the subtraction of the instrument (telescope) polarisation offset and the correction for the zero point of the polarisation angle θ (relative offset angle between polarisation analyser and the polarisation directions on the sky). The first correction is an additive correction and the second is a rotational transformation. These calibration steps require the measurements of polarimetric standard stars, i.e. zero- and highly polarised standard stars. Since instrumental polarisation is wavelength dependent it is necessary to measure the standard stars for all used spectral passbands.

To determine the polarisation offset of the instrument a zero polarisation standard star is observed. After having measured Q and U of the zero polarisation standard star, this is compared to the value from the literature ([Hsu and Breger, 1982], [Whittet et al., 1992], [Turnshek et al., 1990]) and the difference is subtracted from the scientific data.

For determining the zero point of the polarisation angle, highly polarised standard stars must be measured with known polarisation degree p and polarisation direction θ. The measured polarisation direction is then compared to the angle from the literature and a rotational correction is applied to the measured Q and U values.