# Chapter 2

# Light scattering theory: Polarization

*In this Chapter, a brief account on fundamentals of polarization theory has been put forth. It begins with a note on history of polarization discovery and acceptance of transverse-wave nature of light (electromagnetic wave); after which, a detailed derivation of polarization theory has been given.*

## 2.1 Briefing polarization

Polarization generally just means "orientation" . It comes from the Greek word "polos" for the axis of a spinning globe. Wave polarization occurs for vector fields. For electromagnetic waves, the vectors are the orthogonal electric and magnetic fields which vibrate in directions perpendicular to the direction of propagation of wave. Polarization of light (electromagnetic waves) is a property of waves that describes the orientation of their oscillations. By convention, the polarization of light is described by specifying the orientation of the wave's electric field at a point in space over one period of the oscillation and the polarization is perpendicular to the wave's direction of travel. The electric field may be oriented in a single direction (linear polarization), or it may rotate as the wave travels (circular or elliptical polarization). The details will be discussed in section 2.4. In general, most of the light sources emit unpolarized light (with electric and magnetic field vectors in random directions), but there are several ways (reflection, selective absorption, double refraction, cyclotron and synchrotron emission) by which light gets polarized. Scattering of light through a media is one of the ways by which polarization occurs. Light scattered from cometary dust is usually partially polarized. The polarization of the scattered radiation is an important observation which can give information on the nature, shape, structure and sizes of the constituent particles.

## 2.2 Historical note

The investigations of polarized light began with the discovery of the phenomenon of double refraction in calcite crystals (calspar) by Erasmus Bartholinus (1625-1698) in 1669. This was followed by the work of Christian Huygens (1629-1695, founder of wave theory of light), who interpreted double refraction by assuming that in the calspar crystal there is a secondary ellipsoidal wave in addition to the primary spherical wave. In the course of his investigations in 1690, Huygens also discovered that each of the two rays arising from refraction by calcite can be extinguished by passing it through a second calcite crystal if the latter crystal is rotated about the direction of the ray. Christian Huygens was the first to suggest that light was not a scalar quantity based on his work on the propagation of light through crystals. Isaac Newton (1642-1727, founder of corpuscular theory of light) interpreted these phenomena by assuming that the rays have “sides” . After a long time, in 1808, Etienne-Louis Malus (1775-1812), observed that the two images obtained by double refraction through a calspar crystal were extinguished alternately as he rotated the calcite crystal. Malus reported this result but offered no explanation. In 1812, Sir David Brewster (1781-1868) discovered that at a particular angle of incidence (Brewster’s angle) the reflected light viewed through a calcite crystal could be extinguished. Further investigations by Brewster revealed that there was a simple relation between what was to be called the Brewster angle and the refractive index of the glass. The significance of Brewster’s discovery was immediately recognized by his contemporaries.

In Newton’s time, scientists were familiar only with longitudinal waves from their work on the propagation of sound; and it was believed that light “waves”, if they existed, were similar to sound waves. Thus, during the eighteenth century the corpuscular theory of light supported by Newton held sway. But the “transversality" of wave, appeared due to discovery of polarization became a serious objection to the acceptance of the wave theory formulated by Huygens. However, due to Thomas Young (1773-1829), Augustin Jean Fresnel (1788-1827), Dominique Francois Arago (1786-1853) and others' work to solve the problem of interference and diffraction by the use of wave theory gave new life to it. The wave theory was further enhanced when it was used to describe the propagation of polarized light through optically active media.

The wave equation appears in classical optics as a hypothesis. It was accepted because it led to the understanding and description of the propagation, diffraction, interference, and polarization of light. A complete foundation for the wave equation was laid by James Clerk Maxwell's (1831-1879) electrodynamic theory and its experimental confirmation by Heinrich Hertz (1857-1894) in the second half of the nineteenth century.
([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

## 2.3 Physical basis of polarization by scattering

Scattering of light by matter is one of the ways by which light gets polarized. Light scattering is a common phenomenon happening in all media that contains atoms. Matter is composed of discrete electric charges: electrons and protons. When electromagnetic wave meets matter, its elecromagnetic field interacts with the localized electromagnetic field of its constituents; which could be a single electron, an atom or molecule, a solid or liquid particle. The electric charges in the matter are set into oscillatory motion by the electric field of the incident wave. Accelerated electric charges radiate electromagnetic energy in all directions. This secondary radiation is called the radiation 'scattered' by the matter. This newly generated scattered wave strikes neighboring atoms, forcing their electrons into vibrations. These vibrating electrons produce another electromagnetic wave that is once more radiated outward in all directions. In addition to reradiating electromagnetic energy, the excited elementary charges may transform part of the incident electromagnetic energy into other forms (thermal energy, for example), a process called absorption. Scattering and absorption are not mutually independent processes. This absorption and reradiation of light waves causes the light to be scattered about the medium.

This scattering process leads to characteristic polarization (partial) of the scattered light. There are a variety of scattering processes like Thomson scattering, Compton scattering, Rayleigh scattering, fluorescence, or Raman scattering regardless of the different underlying physical mechanisms. This scattered radiation can be observed in any direction and varies with the physical properties of the particle and the scattering direction.
([Bohren and Huffman, 2008], [Born and Wolf, 2000])

## 2.4 Wave equation

To discuss polarized light, we need to investigate first the wave equation and its properties. We therefore begin our study of polarized light with the wave equation.

The macroscopic electromagnetic fields inside matter can be described by the Maxwell equations [Jackson, 1999]

$$\nabla.\vec{D}(\vec{r},t) = \rho(\vec{r},t) \tag{2.1a}$$

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t} \tag{2.1b}$$

$$\nabla.\vec{B}(\vec{r},t) = 0 \tag{2.1c}$$

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t} \tag{2.1d}$$

where $\vec{D}$ is the electric displacement, $\vec{E}$ the electric field, $\vec{B}$ the magnetic induction, $\vec{H}$ the magnetic field, $\rho$ the macroscopic charge density, and $\vec{J}$ the macroscopic current density. From these equations, it is deduced that an electromagnetic wave has orthogonal electric and magnetic fields associated with it, which vibrate in directions perpendicular to the direction of propagation.

For time-harmonic (oscillating) plane monochromatic waves propagating in a homogeneous, linear, isotropic, and non-absorbing medium, the electric ($\vec{E}$) and magnetic ($\vec{H}$) fields are always in phase and oscillating in orthogonal directions with respect to the direction of propagation and each other. The complex-field representation of the waves is

$$\vec{E}(\vec{r},t) = E_0 \exp i(\vec{k}.\vec{r} - \omega t) \tag{2.2a}$$

$$\vec{H}(\vec{r},t) = H_0 \exp i(\vec{k}.\vec{r} - \omega t) \tag{2.2b}$$

where $\vec{r}$ is the radius vector from an arbitrary origin, $\vec{k}$ the wave vector, and $\omega$ the angular frequency of the wave.

In discussion of polarization, it is customary to focus attention on the electric field. The electric field of a sinusoidal electromagnetic wave can be decomposed into orthogonal components, each component having an amplitude and a phase. The phase, referred to a particular position or time tells what part of the cycle the electric field is vibrating in.

In a Cartesian system the components of electric field are-

$$E_x(\vec{r},t) = E_{0x} \exp i(\vec{k}.\vec{r} - \omega t + \delta_x) \tag{2.3a}$$

$$E_y(\vec{r},t) = E_{0y} \exp i(\vec{k}.\vec{r} - \omega t + \delta_y) \tag{2.3b}$$

$$E_z(\vec{r},t) = E_{0z} \exp i(\vec{k}.\vec{r} - \omega t + \delta_z) \tag{2.3c}$$

If we take the direction of propagation of the wave in the z direction, then the real part of electric field components in free space can be described by-

$$E_x(z,t) = \Re\{\hat{x}E_{0x} \exp i(kz - \omega t + \delta_x)\} = \hat{x}E_{0x}\cos(\tau + \delta_x) \tag{2.4a}$$

$$E_y(z,t) = \Re\{\hat{y}E_{0y} \exp i(kz - \omega t + \delta_y)\} = \hat{y}E_{0y}\cos(\tau + \delta_y) \tag{2.4b}$$

where $E_{0x}$ and $E_{0y}$ are the maximum amplitudes, and $\delta_x$ and $\delta_x$ are the phases, respectively and $\tau = kz - \omega t$

## 2.5 Optical polarization

As the field propagates, $E_x(z,t)$ and $E_y(z,t)$ give rise to a resultant vector. This vector describes a locus of points in space, and the curve generated by those points will now be derived.

In order to eliminate propagator $\tau$ between the transverse components of the optical field, we re-write the equations as-

$$\frac{E_x}{E_{0x}} = \cos\tau\cos\delta_x - \sin\tau\sin\delta_x \tag{2.5a}$$
$$\frac{E_y}{E_{0y}} = \cos\tau\cos\delta_y - \sin\tau\sin\delta_y \tag{2.5b}$$

Hence,

$$\frac{E_x}{E_{0x}}\sin\delta_y - \frac{E_y}{E_{0y}}\sin\delta_x = \cos\tau\sin(\delta_y - \delta_x) \tag{2.6a}$$
$$\frac{E_x}{E_{0x}}\cos\delta_y - \frac{E_y}{E_{0y}}\cos\delta_x = \sin\tau\sin(\delta_y - \delta_x) \tag{2.6b}$$

Squaring Eq. (2.6a) and (2.6b) and adding gives

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} - 2\frac{E_x}{E_{0x}}\frac{E_y}{E_{0y}}\cos\delta = \sin^2\delta \tag{2.7}$$

where $\delta = \delta_y - \delta_x$

Eq. (2.7) is recognized as the equation of an ellipse and shows that at any instant of time the locus of points described by the optical field as it propagates in space is an ellipse. This behavior is spoken of as optical polarization, and Eq. (2.7) is called the polarization ellipse.

**Linear polarization:** When the two orthogonal components are in phase, i.e. $\delta = 0°$, the Eq. (2.7) is a straight line. The electric vector oscillates in a straight line as it propagates in space with time. The wave is called linearly polarized.

**Circular polarization:** When the two orthogonal components have same am-

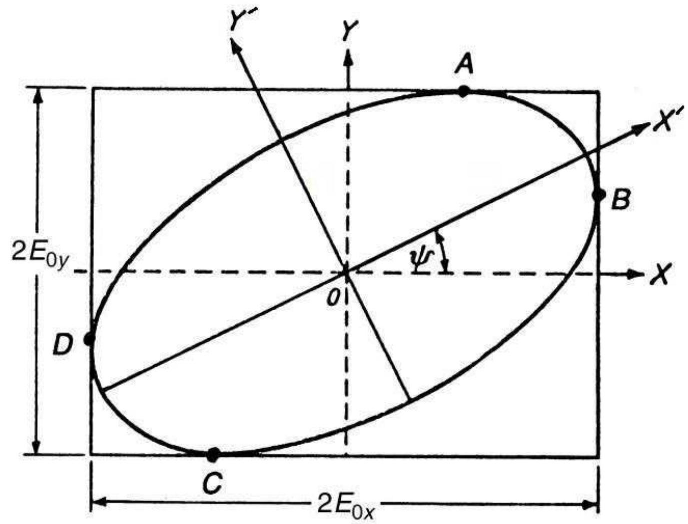


Figure 2.1: Elliptically polarized wave and the polarization ellipse.

plitude ($E_{0x} = E_{0y}$) and the relative phase $\delta = 90°$, the Eq. (2.7) is a circle. The electric vector oscillates (rotates) in a circle as it propagates in space with time. The wave is called circularly polarized.

**Elliptical polarization:** In general, the two components of a electromagnetic wave have arbitrary amplitude and phases. The resultant is the ellipse shown by Eq. (2.7). The resultant electric vector propagates in space with time as an ellipse and the wave is called elliptically polarized.

**Partial polarization:** Light is composed of an ensemble of electromagnetic waves. A group of electromagnetic waves travelling in the same direction can have some linearly polarized waves, some circularly polarized waves, and some elliptically polarized waves. When they are combined, the resulting light can be unpolarized, partially linearly polarized or partially elliptically polarized. Unpolarized light occurs when there are no fixed directions of electric field and also no fixed phase relations between the two orthogonal field components. In general, light is partially polarized and can be decomposed into unpolarized light and elliptically polarized light.

([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

## 2.6 Elliptical parameters of the polarization ellipse

A polarization ellipse is characterized by its ellipticity, the ratio of the length of its semiminor axis to that of the semimajor axis, and its azimuth, the angle between the semimajor axis and an arbitrary reference direction. Handedness, ellipticity, azimuth, together with irradiance, are the elliptical or ellipsometric parameters of a plane wave.

In general, the axes of the ellipse are not in the OX and OY directions. Eq. (2.7) the presence of the "product" term $E_x E_y$ shows that it is actually a rotated ellipse; in the standard form of an ellipse the product term is not present.

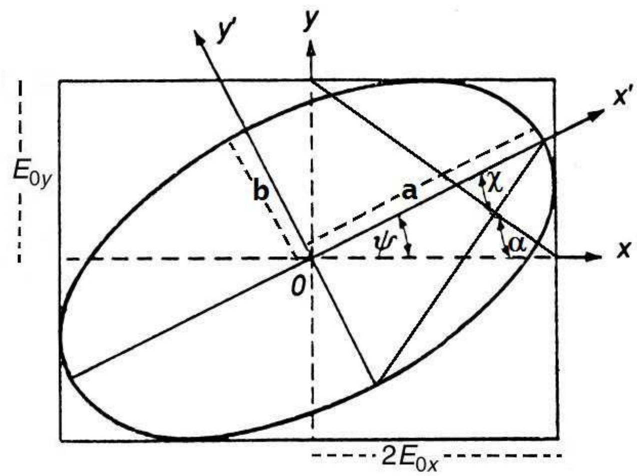


Figure 2.2: The rotated polarization ellipse.

In Fig. (2.2), the rotated ellipse has been shown. Let OX and OY be the initial, unrotated axes, and let OX$'$ and OY$'$ be a new set of axes along the rotated ellipse. Furthermore, let $\psi(0 \leq \psi \leq \pi)$ be the angle between OX and the direction OX$'$ of the major axis.

The components $E_{x'}$ and $E_{y'}$ are

$$E'_x = E_x \cos\psi + E_y \sin\psi \tag{2.8a}$$

$$E'_y = -E_x \sin\psi + E_y \cos\psi \tag{2.8b}$$

If 2a and 2b $(a \geq b)$ are the lengths of the major and minor axes, respectively, then the equation of the ellipse in terms of OX$'$ and OY$'$ can be written as

$$E'_x = a\cos(\tau + \delta') \tag{2.9a}$$

$$E'_y = \pm b\sin(\tau + \delta') \tag{2.9b}$$

where $\tau$ is the propagator and $\delta'$ is an arbitrary phase. The $\pm$ sign describes the two possible senses in which the end point of the field vector can describe the ellipse.

Substituting Eq. (2.4a),(2.4b) and Eq. (2.9a),(2.9b) in Eq. (2.8a),(2.8b) and expanding the terms, we get

$$a(\cos\tau\cos\delta' - \sin\tau\sin\delta') = E_{0x}(\cos\tau\cos\delta_x - \sin\tau\sin\delta_x)\cos\psi + E_{0y}(\cos\tau\cos\delta_y - \sin\tau\sin\delta_y)\sin\psi \tag{2.10a}$$

$$\pm b(\sin\tau\cos\delta' + \cos\tau\sin\delta') = -E_{0x}(\cos\tau\cos\delta_x - \sin\tau\sin\delta_x)\sin\psi + E_{0y}(\cos\tau\cos\delta_y - \sin\tau\sin\delta_y)\cos\psi \tag{2.10b}$$

Equating the coefficients of $\cos\tau$ and $\sin\tau$ leads to the following equations

$$a\cos\delta' = E_{0x}\cos\delta_x\cos\psi + E_{0y}\cos\delta_y\sin\psi \tag{2.11a}$$

$$a\sin\delta' = E_{0x}\sin\delta_x\cos\psi + E_{0y}\sin\delta_y\sin\psi \tag{2.11b}$$

$$\pm b\cos\delta' = E_{0x}\sin\delta_x\sin\psi - E_{0y}\sin\delta_y\cos\psi \tag{2.11c}$$

$$\pm b\sin\delta' = E_{0x}\cos\delta_x\sin\psi - E_{0y}\cos\delta_y\cos\psi \tag{2.11d}$$

Squaring and adding Eq. (2.11a),(2.11b); also Eq. (2.11c),(2.11d) and using $\delta = \delta_y - \delta_x$, we get

$$a^2 = E_{0x}^2 \cos^2\psi + E_{0y}^2 \sin^2\psi + 2E_{0x}E_{0y}\cos\psi\sin\psi\cos\delta \tag{2.12a}$$

$$b^2 = E_{0x}^2 \sin^2\psi + E_{0y}^2 \cos^2\psi - 2E_{0x}E_{0y}\cos\psi\sin\psi\cos\delta \tag{2.12b}$$

Adding Eq. (2.12a) and (2.12b)

$$a^2 + b^2 = E_{0x}^2 + E_{0y}^2 \tag{2.13}$$

Multiplying Eq. (2.11a) by (2.11c) and Eq. (2.11b) by (2.11d) and adding

$$\pm ab = E_{0x}E_{0y}\sin\delta \tag{2.14}$$

Dividing Eq. (2.11d) by (2.11a) and Eq. (2.11c) by (2.11b) and adding

$$(E_{0x}^2 - E_{0y}^2)\sin 2\psi = 2E_{0x}E_{0y}\cos\delta\cos 2\psi \tag{2.15}$$

$$\tan 2\psi = \frac{2E_{0x}E_{0y}\cos\delta}{E_{0x}^2 - E_{0y}^2} \tag{2.16}$$

This relates angle of rotation $\psi$ to $E_{0x}$, $E_{0y}$ and $\delta$. It can be noted that, $\psi$=0 only for $\delta$=$\frac{\pi}{2}$ or $\frac{3\pi}{2}$ . Similarly, $\psi$=0 only if $E_{0x}$ or $E_{0y}$ is equal to 0.

It is useful to introduce an auxiliary angle $\alpha(0 \leq \alpha \leq \frac{\pi}{2})$ for the polarization ellipse defined by

$$\tan\alpha = \frac{E_{0y}}{E_{0x}} \tag{2.17}$$

Using Eq. (2.17) in Eq. (2.16)

$$\tan 2\psi = \frac{2E_{0x}E_{0y}\cos\delta}{E_{0x}^2 - E_{0y}^2} = \frac{2\tan\alpha}{1-\tan^2\alpha}\cos\delta \tag{2.18}$$

$$\tan 2\psi = (\tan 2\alpha)\cos\delta \tag{2.19}$$

Thus for $\delta$=0 or $\pi$, we have $\psi = \pm\alpha$. For $\delta$=$\frac{\pi}{2}$ or $\frac{3\pi}{2}$, we have $\psi$=0, so the angle of rotation is also 0.

Let us define the angle of ellipticity, $\chi$ as

$$\tan\chi = \pm\frac{b}{a} \qquad where \qquad -\frac{\pi}{4} \leq \chi \leq \frac{\pi}{4} \tag{2.20}$$

For linearly polarized light b=0, so $\chi$=0. For circularly polarized light b=a, so $\chi$=$\pm\pi$/4. Thus, Eq. (2.21) describes the extremes of the ellipticity of the polarization ellipse.

Using Eq. (2.13),(2.14),(2.17)

$$\frac{\pm 2ab}{a^2+b^2} = \frac{2E_{0x}E_{0y}\sin\delta}{E_{0x}^2+E_{0y}^2} = \sin 2\alpha \sin\delta \tag{2.21}$$

Using Eq. (2.20) in (2.21)

$$\sin 2\chi = (\sin 2\alpha)\sin\delta \tag{2.22}$$

This relates ellipticity of the polarization ellipse with parameters $E_{0x}, E_{0y}$ and $\delta$ of the polarization ellipse. For $\delta$=$\frac{\pi}{2}$ or $\frac{3\pi}{2}$, Eq. (2.23) becomes $\chi$=$\pm\alpha$

To summarize, the elliptical parameters $E_{0x}, E_{0y}$ and $\delta$ of the polarization ellipse are related to the orientation angle $\psi$ and ellipticity angle $\chi$ by the following equations:

$$\tan 2\psi = (\tan 2\alpha)\cos\delta \qquad 0 \leq \psi \leq \pi \tag{2.23a}$$

$$\sin 2\chi = \sin 2\alpha \sin\delta \qquad -\frac{\pi}{4} \leq \chi \leq \frac{\pi}{4}, \qquad 0 \leq \alpha \leq \frac{\pi}{2} \tag{2.23b}$$

$$a^2 + b^2 = E_{0x}^2 + E_{0y}^2 \tag{2.23c}$$

$$\tan\alpha = \frac{E_{0y}}{E_{0x}} \tag{2.23d}$$

$$\tan\chi = \frac{\pm b}{a} \tag{2.23e}$$

Elimination of the propagator between the transverse components of the optical field led to the polarization ellipse. Analysis of the polarization ellipse showed that for special cases, it led to forms which can be interpreted as linearly polar-

ized light and circularly polarized light. This description of light in terms of the polarization ellipse is very useful because it enables us to describe by means of a single equation various states of polarized light.
([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

## 2.7 Stokes polarization parameters

Representation of light by polarization ellipse is inadequate for several reasons. As the beam of light propagates through space, it forms polarization ellipse of different shapes in a time interval of the order $10^{-15}$ sec. This period of time is clearly too short to allow us to follow the tracing of the ellipse. Another serious limitation is that the polarization ellipse is only applicable to describing light that is completely polarized (linear, circular or elliptical). In nature, light is very often unpolarized or partially polarized. Thus, the polarization ellipse is an idealization of the true behavior of light; it is only correct at any given instant of time. Moreover, although the elliptical parameters completely specify a monochromatic wave of given frequency and are readily visualized, they are difficult to be measured directly (with the exception of irradiance, which can easily be measured with a suitable detector) and are not adaptable to a discussion of partially polarized light. The irradiance of two incoherently superposed beams are additive, but no such additivity exists for the other three elliptical parameters. These limitations force us to consider an alternative description of polarized light in which only observed or measured quantities enter. ([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

In 1852, Sir George Gabriel Stokes (1819-1903) discovered that the polarization behavior could be described by four measurable quantities now known as the Stokes polarization parameters. The first parameter expresses the total intensity of the optical field. The remaining three parameters describe the polarization state. Stokes formulated it in order to provide a suitable mathematical description of the Fresnel-Arago interference laws (1818) based on experiments carried out with an unpolarized light source. He also showed that his parameters could be applied not only to unpolarized light but to partially polarized and completely polarized light as well. ([Stokes, 1852], [Stokes, 1862])

Unfortunately, Stokes' paper was forgotten for nearly a century. Its importance was finally brought to the attention of the scientific community by the Nobel laureate S. Chandrasekhar in 1947, who used the Stokes parameters to formulate the radiative transfer equations for the scattering of partially polarized light. The Stokes parameters have been a prominent part of the optical literature on polarized light ever since. [Chandrasekhar, 1960]

## 2.8 Derivation of Stokes parameters

Let us consider a pair of plane waves (not necessarily monochromatic) that are orthogonal to each other at a point (say z=0) in space, represented by the equations:

$$E_x(t) = E_{0x}(t)\cos[\tau + \delta_x(t)] \tag{2.24a}$$

$$E_y(t) = E_{0y}(t)\cos[\tau + \delta_y(t)] \tag{2.24b}$$

where $E_x(t)$ and $E_y(t)$ are the instantaneous amplitudes, $\omega$ is the instantaneous angular frequency, and $\delta_x(t)$ and $\delta_x(t)$ are the instantaneous phase factors. At all times the amplitudes and phase factors fluctuate slowly compared to the rapid vibrations of the cosinusoids.

The explicit removal of the term $\omega t$ between Eq. (2.24a),(2.24b) yields the familiar polarization ellipse, which is valid, in general, only at a given instant of time

$$\frac{E_x^2(t)}{E_{0x}^2(t)} + \frac{E_y^2(t)}{E_{0y}^2(t)} - 2\frac{E_x(t)}{E_{0x}(t)}\frac{E_y(t)}{E_{0y}(t)}\cos\delta = \sin^2\delta \tag{2.25}$$

where $\delta = \delta_y - \delta_x$

For monochromatic radiation, the amplitudes and phases are constant for all time, so Eq. (2.25) reduces to

$$\frac{E_x^2(t)}{E_{0x}^2} + \frac{E_y^2(t)}{E_{0y}^2} - 2\frac{E_x(t)}{E_{0x}}\frac{E_y(t)}{E_{0y}}\cos\delta = \sin^2\delta \tag{2.26}$$

Where $E_{0x}, E_{0y}$ and $\delta$ are constants.

In order to represent Eq. (2.26) in terms of observables of optical field, we must take an average over the time of observation. Because it is a long period of time relative to the time for a single oscillation, this time can be taken as infinite.

We write Eq. (2.26) as

$$\frac{< E_x^2(t) >}{E_{0x}^2} + \frac{< E_y^2(t) >}{E_{0y}^2} - 2\frac{< E_x(t)E_y(t) >}{E_{0x}E_{0y}} \cos\delta = \sin^2\delta \tag{2.27}$$

where

$$< E_i(t)E_j(t) >= \lim_{T\to\infty} \frac{1}{T} \int_0^T E_i(t)E_j(t)dt \qquad i, j = x, y \tag{2.28}$$

Multiplying Eq. (2.27) by $4E_{0x}^2 E_{0y}^2$,

$$4E_{0y}^2 < E_x^2(t) > + 4E_{0x}^2 < E_y^2(t) > - 8E_{0x}E_{0y} < E_x(t)E_y(t) > \cos\delta = (2E_{0x}E_{0y}\sin\delta)^2 \tag{2.29}$$

Using Eq. (2.28), we find the average values as

$$< E_x^2(t) >= \frac{1}{2}E_{0x}^2 \tag{2.30a}$$

$$< E_y^2(t) >= \frac{1}{2}E_{0y}^2 \tag{2.30b}$$

$$< E_x(t)E_y(t) >= \frac{1}{2}E_{0x}E_{0y}\cos\delta \tag{2.30c}$$

Substituting Eq. (2.30a),(2.30b),(2.30c) in Eq. (2.29)

$$2E_{0x}^2 E_{0y}^2 + 2E_{0x}^2 E_{0y}^2 - (2E_{0x}E_{0y}\cos\delta)^2 = (2E_{0x}E_{0y}\sin\delta)^2 \tag{2.31}$$

Add and subtract the quantity $(E_{0x}^4 + E_{0y}^4)$

$$(E_{0x}^2 + E_{0y}^2)^2 - (E_{0x}^2 - E_{0y}^2)^2 - (2E_{0x}E_{0y}\cos\delta)^2 = (2E_{0x}E_{0y}\sin\delta)^2 \quad (2.32)$$

We now write the quantities inside the parentheses as

$$S_0 = E_{0x}^2 + E_{0y}^2 \quad (2.33a)$$

$$S_1 = E_{0x}^2 - E_{0y}^2 \quad (2.33b)$$

$$S_2 = 2E_{0x}E_{0y}\cos\delta \quad (2.33c)$$

$$S_3 = 2E_{0x}E_{0y}\sin\delta \quad (2.33d)$$

Eq. (2.32) takes the form,

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \quad (2.34)$$

It can be shown using Schwarz's [Sopka, 1972] inequality that for any state of polarized light the Stokes parameters always satisfy the relation

$$S_0^2 \geqslant S_1^2 + S_2^2 + S_3^2 \quad (2.35)$$

The equality sign is valid for completely polarized light and inequality for partially polarized light or unpolarized light.

In terms of orientation angle $\psi$ and the ellipticity angle $\chi$ of the polarization ellipse

$$\tan 2\psi = \frac{S_2}{S_1} \quad (2.36)$$

$$\sin 2\chi = \frac{S_3}{S_0} \quad (2.37)$$

The four equations (Eq. (2.33a),(2.33b),(2.33c),(2.33d)) are the Stokes polarization parameters for a plane wave introduced into optics by Sir George Gabriel Stokes in 1852. The Stokes parameters are real quantities, and they are simply the observables of the polarization ellipse and, hence, the optical field.

$S_0$ is the total intensity of the light.

$S_1$ describes the amount of linear horizontal or vertical polarization.

$S_2$ describes the amount of linear $+45°$ or $-45°$ polarization.

$S_3$ describes the amount of right or left circular polarization contained within the beam.

([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

## 2.9 Stokes parameters in complex form

To obtain the Stokes parameters of an optical beam, it is easier to take a time average of the polarization ellipse in terms of complex amplitudes.

$$E_x(t) = E_{0x}exp[i(\omega t + \delta_x)] = E_x exp(i\omega t) \tag{2.38a}$$

$$E_y(t) = E_{0y}exp[i(\omega t + \delta_y)] = E_y exp(i\omega t) \tag{2.38b}$$

where $E_x = E_{0x}exp(i\delta_x)andE_y = E_{0y}exp(i\delta_y)$ are complex amplitudes.

The Stokes parameters for a plane wave are now obtained from the formulae-

$$S_0 =< E_xE_x^* > + < E_yE_y^* > \tag{2.39a}$$

$$S_1 =< E_xE_x^* > - < E_yE_y^* > \tag{2.39b}$$

$$S_2 =< E_xE_y^* > + < E_yE_x^* > \tag{2.39c}$$

$$S_3 = i(< E_xE_y^* > - < E_yE_x^* >) \tag{2.39d}$$

([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

## 2.10 The Stokes vectors

The four Stokes parameters can be arranged in a column matrix and written as

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{2.40}$$

The Eq. (2.40) is called the Stokes vector or Stokes column matrix. Mathematically, it is not a vector, but through custom it is called a vector.

The Stokes vector for elliptically polarized light is then

$$S = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix} = \begin{pmatrix} < E_x E_x^* > + < E_y E_y^* > \\ < E_x E_x^* > - < E_y E_y^* > \\ < E_x E_y^* > + < E_y E_x^* > \\ i(< E_x E_y^* > - < E_y E_x^* >) \end{pmatrix} \tag{2.41}$$

Eq. (2.41) is also called the Stokes vector for a plane wave. The Stokes vectors for linearly and circularly polarized light are readily found from (2.41).

Thus, the Stokes parameters are a logical consequence of the wave theory. They give a complete description of any polarization state of light and those quantities that are measured. Originally, the Stokes parameters were used only to describe the measured intensity and polarization state of the optical field. But by forming the Stokes parameters in terms of a column matrix, the so-called Stokes vector, we are led to a formulation in which we obtain not only measurable but also observable, which can be seen in polarimetric experiments. As a result, the formalism of the Stokes parameters is far more versatile than originally envisioned and possesses a greater usefulness than is commonly known.
([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])

## 2.11 Polarization in terms of Stokes parameters

The Stokes parameters enable us to describe the degree of polarization P for any state of polarization. By definition,

$$P = \frac{I_{pol}}{I_{tot}} = \frac{(S_1^2 + S_2^2 + S_3^2)^{1/2}}{S_0} \qquad 0 \leq P \leq 1 \tag{2.42}$$

where $I_{pol}$ is the intensity of the sum of the polarization components and $I_{tot}$ is the total intensity of the beam.

The value of P=1 corresponds to completely polarized light.

P=0 corresponds to unpolarized light.

And $0 < P < 1$ corresponds to partially polarized light.

([Bohren and Huffman, 2008], [Born and Wolf, 2000], [Goldstein, 2016])